

LEVEL ONE ALGEBRAIC CUSP FORMS OF CLASSICAL GROUPS OF SMALL RANKS

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ABSTRACT. We determine the number of level 1, self-dual, half-algebraic regular, cuspidal automorphic representations of $\mathrm{GL}(n)$ over \mathbb{Q} of any given infinitesimal character, and essentially all $n \leq 8$. For this, we compute the dimensions of spaces of level 1 automorphic forms for certain semisimple \mathbb{Z} -forms of the compact groups $\mathrm{SO}(7)$, $\mathrm{SO}(8)$, $\mathrm{SO}(9)$ (and G_2) and determine Arthur's endoscopic partition of these spaces in all cases. We also give applications to the 121 even lattices of rank 25 and determinant 2 found by Borchers, to level one self-dual automorphic representations of $\mathrm{GL}(n)$ with trivial infinitesimal character, and to vector valued Siegel modular forms of genus 3. A part of our results are conditional.

1. INTRODUCTION

Let $n \geq 2$ be an integer and consider the cuspidal automorphic representations π of $\mathrm{GL}(n)$ over \mathbb{Q} such that :

- (i) (self-duality) $\pi^\vee \simeq \pi$,
- (ii) (conductor 1) π_p is unramified for each prime p ,
- (iii) (algebraicity) π_∞ is half-algebraic regular.

Our main aim in this paper is to give for small values of n , namely all $n \leq 8$, the number of such representations as a function of the infinitesimal character of π_∞ .

We first need to say a bit more about the meaning of condition (iii). According to Langlands parameterization, π_∞ is uniquely determined by its Langlands parameter, which is a conjugacy class of semisimple continuous group homomorphisms

$$L(\pi_\infty) : W_{\mathbb{R}} \rightarrow \mathrm{GL}_n(\mathbb{C}).$$

Here $W_{\mathbb{R}}$ is the Weil group¹ of \mathbb{R} . Let $\theta : W_{\mathbb{C}} \rightarrow \mathbb{C}^*$ be the unitary character $\theta(z) = (\frac{z}{|z|})^{\frac{1}{2}} := \frac{z}{|z|}$, and for $w \geq 0$ an integer let $I_w = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \theta^w$. Condition (iii) means that there are $[n/2]$ distinct integers²

$$w_1 > w_2 > \cdots > w_{[n/2]} \geq 0$$

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¹Recall that $W_{\mathbb{R}}$ is a non-split extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $W_{\mathbb{C}} = \mathbb{C}^*$, for the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on \mathbb{C}^* . It contains an element $j \in W_{\mathbb{R}} \setminus W_{\mathbb{C}}$ such that $j^2 = -1 \in \mathbb{C}^*$.

² $[x]$ denotes the floor of the real number x .

such that :

- (iii, n even) $L(\pi_\infty) \simeq \bigoplus_{i=1}^{n/2} I_{w_i}$,
- (iii, n odd) $L(\pi_\infty) \simeq \chi \oplus \bigoplus_{i=1}^{[n/2]} I_{w_i}$ where χ is an order 2 character of $W_{\mathbb{R}}$, and where furthermore $w_{[n/2]} > 0$.

We shall call the integer $w(\pi) = w_1$ the (*effective*) *motivic weight* of π , and the w_i the *Hodge numbers* of π . It follows from the work of Arthur (see Cor. 3.10) that $w_i \equiv w(\pi) \pmod{2}$ for all i , and that $w(\pi)$ is even if n is odd. If $w(\pi)$ is odd, the self-dual representation $L(\pi_\infty)$ preserves a non-degenerate symplectic pairing and no symmetric one, thus π will be called *symplectic*. Similarly, if $w(\pi)$ is even then $L(\pi_\infty)$ preserves a non-degenerate symmetric pairing and no symplectic one, and π will be called *orthogonal*.

Recall that the group $W_{\mathbb{R}}$ has exactly two order 2 characters, the trivial 1 and the sign ε , so that $I_0 = 1 \oplus \varepsilon$. The global central character of a π satisfying (i) and (ii) is trivial by the structure of the idèles of \mathbb{Q} , hence so is $\det(L(\pi_\infty))$. As $\det(I_w) = \varepsilon^{w+1}$, it follows that the character χ occurring in the case n odd is uniquely determined by n , namely $\chi = \varepsilon^{[n/2]}$. Moreover, if n is even and π is orthogonal, then $n \equiv 0 \pmod{4}$.

Definition 1.1. *Let r and $w_1 > \dots > w_r \geq 0$ be integers. We denote by $S(w_1, \dots, w_r)$ (resp. $O(w_1, \dots, w_r)$ and $O^*(w_1, \dots, w_r)$) the number of cuspidal automorphic representations of $\mathrm{GL}(2r)$ (resp. $\mathrm{GL}(2r)$ and $\mathrm{GL}(2r+1)$) satisfying (i), (ii) and (iii) above with Hodge numbers $w_1 > \dots > w_r$, and which are symplectic (resp. orthogonal).*

Standard finitude results proved by Harish-Chandra ensure that those numbers are indeed finite. As explained above, the central character of such a π is necessarily trivial, so that one can actually replace $\mathrm{GL}(\ast)$ by $\mathrm{PGL}(\ast)$ in this definition. The main question addressed in this paper is the following :

Problem 1.2. *For any integer $r \geq 1$ and any $w_1 > \dots > w_r \geq 0$, give an explicit formula for $S(w_1, \dots, w_r)$, $O(w_1, \dots, w_r)$ and $O^*(w_1, \dots, w_r)$.*

There are several motivations for this. A first one is the well-known problem of finding the smallest integer $n \geq 1$ such that the cuspidal cohomology $H_{\mathrm{cusp}}^*(\mathrm{SL}_n(\mathbb{Z}), \mathbb{Q})$ does not vanish. It would be enough to find an integer $n \geq 1$ such that $S(n-1, \dots, 5, 3, 1) \neq 0$ if n is even, or such that $O^*(n-1, \dots, 4, 2) \neq 0$ if n odd. Results of Mestre [M], Fermigier [Fe] and Miller [Mi] ensure that such an n has to be ≥ 27 (although those works do not assume the self-duality condition). We shall go back to these questions at the end of this introduction.

As a second motivation, it follows from Arthur's endoscopic classification [A3] that the dimension of various spaces of modular forms for classical reductive groups over \mathbb{Z} have a "simple" expression in terms of these numbers. This includes holomorphic Siegel modular forms for $\mathrm{Sp}(2g, \mathbb{Z})$ and level 1 algebraic automorphic forms for the \mathbb{Z} -forms of $\mathrm{SO}(p, q)$ which are semisimple over \mathbb{Z} (such twisted forms exist when $p - q \equiv 0, \pm 1 \pmod{8}$). We will say much more about this later as this is the main theme of this paper.

A third motivation is the deep conjectural relation (due to Langlands) that those numbers S , O and O^* share with arithmetic geometry and \mathbb{Q} -motives over \mathbb{Z} . Let us

state it by the mean of ℓ -adic Galois representations of the absolute Galois \mathbb{Q} . Fix a prime ℓ , algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_\ell$ of \mathbb{Q} and \mathbb{Q}_ℓ , as well as a pair $\iota = (\iota_\infty, \iota_\ell)$ of fields embeddings $\iota_\infty : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ and $\iota_\ell : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_\ell$. Let π be a cuspidal automorphic representation of $\mathrm{GL}(n)$ over \mathbb{Q} satisfying (i), (ii) and (iii). Let

$$\pi^{\mathrm{alg}} = \pi \otimes |\det|^{\frac{-w(\pi)}{2}},$$

so that $\pi^{\mathrm{alg}, \vee} = \pi^{\mathrm{alg}} \otimes |\det|^{w(\pi)}$. It is known that for each prime p the Satake parameter of π_p^{alg} , a semisimple conjugacy class in $\mathrm{GL}_n(\mathbb{C})$, has its characteristic polynomial $P(\pi_p)$ with coefficients in $\iota_\infty(\overline{\mathbb{Z}})$. By works of many authors, there is a unique continuous semisimple Galois representation

$$\rho_{\pi, \iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \begin{cases} \mathrm{GSp}(n, \overline{\mathbb{Q}}_\ell) & \text{if } \pi \text{ is symplectic,} \\ \mathrm{GO}(n, \overline{\mathbb{Q}}_\ell) & \text{if } \pi \text{ is orthogonal,} \end{cases}$$

which is unramified outside $\{\infty, \ell\}$ and such that for each prime $p \neq \ell$, the characteristic polynomial of a geometric Frobenius element at $p \neq \ell$ is $\iota_\ell \iota_\infty^{-1}(P(\pi_p))$. This representation $\rho_{\pi, \iota}$ satisfies the following extra properties :

- (i) The restriction of $\rho_{\pi, \iota}$ to a decomposition group at ℓ is crystalline³ with Hodge-Tate numbers⁴ the $\frac{\pm w_i + w(\pi)}{2}$, $i = 1, \dots, [n/2]$, plus $\frac{w(\pi)}{2}$ if n is odd.
- (ii) The trace of the complex conjugations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ belongs to $\{-1, 0, 1\}$.
- (iii) The similitude factor of $\rho_{\pi, \iota}$ is $\omega_\ell^{-w(\pi)}$, where ω_ℓ is the ℓ -adic cyclotomic character.

Property (iii) actually follows from the other ones. Moreover, observe that the Hodge-Tate numbers in (i) are integers between 0 and $w(\pi)$, those two included, and are distinct except when n is even, π is orthogonal, and $w_{n/2} = 0$, in which case $\frac{w(\pi)}{2}$ occurs exactly twice (but note that the trace of the complex conjugation is 0 by (iii)). The representation $\rho_{\pi, \ell}$ is conjecturally irreducible but this is only known for $n \leq 5$.

A standard conjecture of Langlands in the lead of Shimura, Taniyama, and Weil, predicts that conversely any abstract Galois representation with all these properties should come from one and only one π satisfying (i), (ii) and (iii), so that Problem 1.1 is conjecturally equivalent to counting certain ℓ -adic Galois representations.

Conjecture 1.3. (*Langlands*) *The number of isomorphism classes of irreducible continuous representations*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GSp}(n, \overline{\mathbb{Q}}_\ell) \quad (\text{resp. } \mathrm{GO}(n, \overline{\mathbb{Q}}_\ell))$$

which are unramified outside ℓ , and which satisfy (i), (ii) and (iii) above for the integers $w_1 > \dots > w_{[n/2]} \geq 0$ and for $w(\pi) = w_1$, is⁵ $S(w_1, \dots, w_{[n/2]})$ (resp. $O(w_1, \dots, w_{[n/2]})$ if n is even, $O^(w_1, \dots, w_{[n/2]})$ if n is odd).*

³At present, this crystalline property is only known under the extra assumption $w_{[n/2]} \neq 0$.

⁴Our convention is that the Hodge-Tate number of the cyclotomic character is -1 .

⁵Included in this statement is the conjecture that for such a symplectic Galois representation the motivic weight w_1 is necessarily odd. In the orthogonal case, it is even by assumption (i).

An especially interesting case is when one can show that some $X(w_1, \dots, w_r)$ is zero, where $X \in \{S, O, O^*\}$, which thus translates to a conjectural non-existence theorem about Galois representations. An important source of geometric Galois representations satisfying conditions close to (i), (iii) and (iii) is the ℓ -adic étale cohomology of proper smooth schemes over \mathbb{Z} , about which solving Problem 1.1 would shed interesting lights. A famous result in this style is the proof by Abrashkin and Fontaine that there are no abelian scheme over \mathbb{Z} (hence no projective smooth curve over \mathbb{Z} of nonzero genus), which had been conjectured by Shaffarevich. The corresponding vanishing statement about cuspidal automorphic forms had been previously checked by Mestre and Serre (see [M]). See also Khare's paper [Kh] for other conjectures in this spirit as well as a discussion about the applications to the generalized Serre's conjecture.

Our first main result is the following. As many results that we prove in this paper, it depends on the fabulous work of Arthur in [A3]. As explained loc. cit. this work is still conditional to two assumptions, namely the stabilization of the twisted trace formula and a collection of twisted character identities for $\mathrm{GL}(n, \mathbb{R})$. All the results below depending on these assumptions will be marked by a simple star *. We shall also need to use certain results concerning inner forms of classical groups which have been announced by Arthur (see [A3, Chap. 9]) but which are not yet available or even precisely stated. We have thus formulated the precise general results that we expect in two explicit conjectures 3.19 and 3.23. Those conjectures include actually a bit more than what has been announced by Arthur in [A3], namely also the standard expectation that for Adams-Johnson archimedean Arthur parameters, there is an identification between Arthur's packets in [A3] and the ones of Adams and Johnson in [AdJ]. The precise special cases that we need are detailed in Chapter 3.22. All the results below depending on the assumptions of [A3] as well as on the assumptions 3.19 and 3.23 will be marked by a double star **. Of course, the tremendous recent progresses in this area, such as the proof by Laumon, Ngo and Waldspurger of Langlands' fundamental lemma, Arthur's work [A3], or the work of Shelstad [Sh2], allow some optimism about the future of all these assumptions !

Theorem 1.4.** *Problem 1 is solved for S when $r \leq 4$, for O when $r \leq 4$, assuming furthermore that $w_4 \neq 0$ when $r = 4$, and for O^* if $r \leq 2$. Moreover, we have an explicit formula for $2 \cdot O(w, v, u, 0) + O^*(w, v, u)$ for any even $w > v > u > 0$.*

Although our formulae are explicit, one cannot write them down here as they are much too big, see e.g. below for some information about the formula for $S(w, v, u)$. On the other hand, we implemented them on a computer and have a program which, in those cases, takes any (w_1, \dots, w_r) and gives $X(w_1, \dots, w_r)$. We will discuss the results in each particular case later, see also the website [ChR] for some data. We shall go back to this later but let us already mention that the computations in [BFVdG] also allow to compute $O^*(w_1, w_2, w_3)$, hence $O(w_1, w_2, w_3, 0)$ as well by the above result. In particular, we have now explicit formulas for the number of symplectic (resp. orthogonal) π satisfying (i), (ii) and (iii) of any given infinitesimal character in any rank $n \leq 8$.

Let us start with some simple examples. First, a standard translation ensures that for each odd integer $w \geq 1$, then $S(w)$ is the dimension of the space $S_{w+1}(\mathrm{SL}(2, \mathbb{Z}))$ of cusp forms of weight $w + 1$ for the full modular group $\mathrm{SL}(2, \mathbb{Z})$. We therefore have the well-known formula

$$(1) \quad S(w) = \dim S_{w+1}(\mathrm{SL}(2, \mathbb{Z})) = \left[\frac{w+1}{12} \right] - \delta_{w \equiv 1 \pmod{12}} \cdot \delta_{w > 1}$$

where δ_P is 1 if property P holds and 0 otherwise.

The next symplectic case is to give $S(w, v)$ for $w > v$ odd positive integers. This case, which is no doubt well known to the experts, may be deduced from Arthur's results [A3] and a computation by R. Tsushima [T]. Let $S_{(w,v)}(\mathrm{Sp}(4, \mathbb{Z}))$ be the space of vector-valued Siegel modular forms of genus 2 for the coefficient systems $\mathrm{Sym}^j \otimes \det^k$ where $j = v - 1$ and $k = \frac{w-v}{2} + 2$ (we follow the conventions in [VdG, §25]). Using the geometry of the Siegel threefold, Tsushima was able to give an explicit formula for $\dim S_{(w,v)}(\mathrm{Sp}(4, \mathbb{Z}))$ in terms of (w, v) . This formula is already too big to give it here, but see loc. cit. Thm. 4.⁶ An examination of Arthur's results for the Chevalley group $\mathrm{SO}(3, 2) = \mathrm{PGSp}(4)$ over \mathbb{Z} shows then that

$$(2) \quad S(w, v) = \dim S_{(w,v)}(\mathrm{Sp}(4, \mathbb{Z})) - \delta_{v=1} \cdot \delta_{w \equiv 1 \pmod{4}} \cdot S(w).$$

The term which is subtracted is actually the dimension of the Saito-Kurokawa subspace of $S_{(w,v)}(\mathrm{Sp}(4, \mathbb{Z}))$. That this is the only term to subtract is explained by Arthur's multiplicity formula, and we shall go back to this later. We refer to Table 6 for the first nonzero values of $S(w, v)$. It follows for instance that for $w \leq 23$, then $S(w, v) = 0$ unless (w, v) is in the following list :

$$(19, 7), (21, 5), (21, 9), (21, 13), (23, 7), (23, 9), (23, 13)$$

In all those cases $S(w, v) = 1$. Moreover the first w such that $S(w, 1) \neq 0$ is $w = 37$.

Our first serious contribution is the computation of $S(w_1, \dots, w_r)$ for $r = 3$ and 4 (and any w_i). Our strategy is to compute first the dimension of the spaces of level 1 automorphic forms for two certain orthogonal groups $\mathrm{SO}(7)$ and $\mathrm{SO}(9)$ which are reductive over \mathbb{Z} . These groups are respectively the special orthogonal groups of the root lattice of the complex Lie algebras E_7 and $E_8 \times A_1$. Those groups have compact real points $\mathrm{SO}(n, \mathbb{R})$ for $n = 7$ and 9 and both have class number 1, so that we are reduced to determine the dimension of the invariants of their integral points, namely the positive Weyl group of E_7 and the Weyl group of E_8 , in any given finite dimensional irreducible representation of the corresponding $\mathrm{SO}(n, \mathbb{R})$. We will say more about this computation later. The second important step is to rule out all the endoscopic or non-tempered contributions predicted by Arthur's forthcoming last chapter of [A3] for those groups to get the exact values of $S(*)$. These contributions are given by a detailed study of Arthur's multiplicity formula. We actually explicitly describe this formula in several important cases, namely the general case of a semisimple \mathbb{Z} -group G with $G(\mathbb{R})$ compact, and also later for the contribution of holomorphic discrete series to the level 1 discrete spectrum

⁶There is a much simpler closed formula for the Poincaré series of the $S(w, 1)$ due to Igusa : see [VdG, §9].

of $\mathrm{Sp}(2g)$. In the cases of $\mathrm{SO}(7)$ (resp. $\mathrm{SO}(9)$) there are for instance 9 (resp. 16) multiplicity formulas to determine. They require in particular the computations of S , O and O^* of smaller ranks : we refer to Chapters 5 and 6 for the complete study.

We refer to Tables 7 and 8 for the first non zero values of $S(w_1, \dots, w_r)$ for $r = 3, 4$, and to the url [ChR] for much more data. Here is a small sample of our results.

Corollary 1.5.** (i) $S(w_1, w_2, w_3)$ vanishes for $w_1 < 23$.

(ii) There are exactly 7 triples (w_1, w_2, w_3) with $w_1 = 23$ such that $S(w_1, w_2, w_3)$ is nonzero :

$$(23, 13, 5), (23, 15, 3), (23, 15, 7), (23, 17, 5), (23, 17, 9), (23, 19, 3), (23, 19, 11),$$

and for all of them $S(w_1, w_2, w_3) = 1$.

Corollary 1.6.** (i) $S(w_1, w_2, w_3, w_4)$ vanishes for $w_1 < 25$.

(ii) There are exactly 33 triples (w_2, w_3, w_4) such that $S(25, w_2, w_3, w_4) \neq 0$ and for all of them $S(25, w_2, w_3, w_4) = 1$, except $S(25, 21, 15, 7) = S(25, 23, 11, 5) = 2$ and $S(25, 23, 15, 5) = 3$.

We discuss in § 4.19 the conjectural (Langlands-)Sato-Tate groups of the automorphic π satisfying (i), (ii) and (iii) in dimension $n \leq 8$. The Sato-Tate group of each of the 7 automorphic representations of Cor. 1.5 (ii) (resp. of the 37 automorphic representations of Cor. 1.6 (ii)) turns out to be the compact symplectic group of rank 3 (resp. 4). We have in our database other numerical results concerning the special orthogonal \mathbb{Z} -group $\mathrm{SO}(15)$ of the root lattice of $E_7 \times E_8$. We shall however not present those results here but we hope to add them soon to the database [ChR].

Let us discuss now the orthogonal case. We have already seen that $O(w_1, \dots, w_r)$ vanishes for r odd. Another general fact is the following.

Proposition* 1.7. If $\frac{1}{2}(\sum_{i=1}^r w_i) \not\equiv [\frac{r+1}{2}] \pmod{2}$ then

$$O(w_1, \dots, w_r) = O^*(w_1, \dots, w_r) = 0.$$

Proof — Indeed, recall from [Ta] that the epsilon factor with respect to $e^{2i\pi x}$ of the representation I_w of $W_{\mathbb{R}}$ is i^{w+1} (the choice $w \geq 0$ is important here). The ones of 1, ε are 1 and i respectively. It follows that the global ε -factor of a π satisfying (i), (ii) and (iii) is (see §3.20)

$$\varepsilon(\pi) = \begin{cases} (-1)^{\frac{\sum_{j=1}^{[n/2]} (w_j+1)}{2}} & \text{if } n \not\equiv 3 \pmod{4}, \\ -(-1)^{\frac{\sum_{j=1}^{[n/2]} (w_j+1)}{2}} & \text{otherwise.} \end{cases}$$

We conclude by a result of Arthur [A3, Thm. 1.5.3] that says that an orthogonal π has a trivial global epsilon factor $\varepsilon(\pi)$. \square

Each time we shall write $O(w_1, \dots, w_r)$ and $O^*(w_1, \dots, w_r)$ we shall thus assume from now on that $\frac{1}{2}(\sum_i w_i) \equiv [\frac{r+1}{2}] \pmod{2}$. Here are those numbers for $r \leq 2$.

- Theorem* 1.8.** (i) $O^*(w) = S(\frac{w}{2})$,
 (ii) $O(w, v) = S(\frac{w+v}{2}) \cdot S(\frac{w-v}{2})$ if $v \neq 0$, and $O(w, 0) = \frac{S(w/2) \cdot (S(w/2)-1)}{2}$,
 (iii) $O^*(w, v) = S(\frac{w+v}{2}, \frac{w-v}{2})$.

These identities actually correspond to some simple cases of Langlands functoriality related to the exceptional isogenies $\mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SO}(3, \mathbb{C})$ (*symmetric square*), $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SO}(4, \mathbb{C})$ (*tensor product*) and $\mathrm{Sp}(4, \mathbb{C}) \rightarrow \mathrm{SO}(5, \mathbb{C})$ (*reduced exterior square*). As we shall show, they are all consequences of Arthur's work : see Chapter 4. Another tool in our proof is a general, elementary, lifting result for isogenies between Chevalley groups over \mathbb{Z} . In the language of the standard Langlands conjectures, it asserts that the Langlands group of \mathbb{Z} (a compact connected topological group) is simply connected.

Our main contribution in the orthogonal case is thus the assertion in Thm. 1.4 about $O(w_1, w_2, w_3, w_4)$. We argue as before by considering this time the orthogonal group SO_{E_8} over \mathbb{Z} , which also has class number 1 as E_8 is the unique even unimodular lattice in rank 8. As already said, the precise numbers that we compute are the $O(w_1, w_2, w_3, w_4)$ when $w_4 \neq 0$, as well as the numbers

$$2 \cdot O(w_1, w_2, w_3, 0) + O^*(w_1, w_2, w_3).$$

When this latter number is ≤ 1 , it is thus necessarily equal to $O^*(w_1, w_2, w_3)$, which leads first to the following partial results for the orthogonal representations π of in dimension $n = 7$. See Table 10 and [ChR] for more results.

- Corollary** 1.9.** (i) $O^*(w_1, w_2, w_3)$ vanishes for $w_1 < 24$.
 (ii) There are exactly 8 triples (w_1, w_2, w_3) with $w_1 \leq 26$ such that $O^*(w_1, w_2, w_3) \neq 0$, namely

$$(24, 16, 8), (26, 16, 10), (26, 20, 6), (26, 20, 10), (26, 20, 14),$$

$$(26, 24, 10), (26, 24, 14), (26, 24, 18),$$

in which cases $O^*(w_1, w_2, w_3) = 1$.

Observe that our approach does not allow to tackle this case directly as there is no semisimple \mathbb{Z} -group of type C_3 with compact real points (and actually of type C_l for any $l \geq 3$). On the other hand, our results allow to compute in a number of cases the dimension of the space $S_{w_1, w_2, w_3}(\mathrm{Sp}(6, \mathbb{Z}))$ of vector valued Siegel modular forms of whose infinitesimal character has eigenvalues $\pm w_1, \pm w_2, \pm w_3, 0$ in $\mathfrak{so}(7, \mathbb{C})$. Indeed, we deduce from Arthur's multiplicity formula that :

- Proposition** 1.10.** $\dim S_{w_1, w_2, w_3}(\mathrm{Sp}(6, \mathbb{Z})) = O^*(w_1, w_2, w_3) + O(w_1, w_3) \cdot O^*(w_2)$
 $+ \delta_{w_2 \equiv 0 \pmod{4}} \cdot (\delta_{w_2 = w_3 + 2} \cdot S(w_2 - 1) \cdot O^*(w_1) + \delta_{w_1 = w_2 + 2} \cdot S(w_2 + 1) \cdot O^*(w_3)).$

In particular, the results already stated in this introduction allow to determine the dimension of $S_{w_1, w_2, w_3}(\mathrm{Sp}(6, \mathbb{Z}))$ when $w_1 \leq 26$. We refer to Chapter 9 for more about this and to the website [ChR] for some results. We actually explain in this chapter how to compute *for any genus g* the dimension of the space of Siegel cusp forms for $\mathrm{Sp}(2g, \mathbb{Z})$ of any given regular infinitesimal character in terms of various numbers $S(-)$, $O(-)$ and $O^*(-)$.

The problem of the determination of $\dim S_{w_1, w_2, w_3}(\mathrm{Sp}(6, \mathbb{Z}))$ has been solved by Tsuyumine in [Ts] when $w_1 - w_3 = 4$ (scalar valued Siegel modular forms of weight $k = \frac{1}{2}(w_1 + 2)$). It has also been studied recently in general by Van der Geer, Bini, Bergström and Faber, see e.g. [BFVdG] for the latest account of their beautiful results. They study the moduli spaces of smooth curves of genus three and of principally polarized abelian varieties of dimension 3, both over \mathbb{Z} . In this last paper, the authors give in particular a (partly conjectural) table for certain values of $\dim S_{w_1, w_2, w_3}(\mathrm{Sp}(6, \mathbb{Z}))$: see Table 1 loc. cit. We checked that this table fits our results. As already said, the combination of their results and ours lead to a determination of each $O(w_1, w_2, w_3, 0)$ as well. Let us mention that those authors not only compute dimensions but also certain Hecke eigenvalues.

There is also a funny game we can play with the reductive group G_2 over \mathbb{Z} such that $G_2(\mathbb{R})$ is compact. Indeed, the triples (w, v, u) with $w = v + u$ are related to G_2 and triality for $\mathrm{PGSO}_{\mathrm{E}_8}$. For dimension's reasons, it follows that the first three π 's given by the corollary (ii) above should come by Langlands functoriality from the dual embedding $G_2(\mathbb{C}) \rightarrow \mathrm{SO}(7, \mathbb{C})$. We checked this in a different manner by computing the dimension of the spaces of level 1 automorphic forms for this \mathbb{Z} -group G_2 . Better, assuming an extension of Arthur's work to G_2 , which should follow from the theory of endoscopy, we computed as well the conjectural number

$$G_2(v, u)$$

of rank 7 orthogonal π 's with Hodge numbers $v + u > v > u$ coming dually from the embedding above of G_2 as an explicit function of v and u . They match with the previous corollary, and give a conjectural minoration of $O^*(v + u, v, u)$ in general. We refer to Table 11 for a sample of results. This also confirms certain similar predictions in [BFVdG].

Modular forms of level one for the Chevalley group of type G_2 , and whose archimedean component is a quaternionic discrete series, have been studied by Gan, Gross and Savin in [GaGrS]. They define a notion of Fourier coefficients for those modular forms and give interesting examples of Eisenstein series and of two exceptional theta series coming from the modular forms of level 1 and trivial coefficient of the anisotropic form of F_4 over \mathbb{Q} . Table 11 suggests that the first cusp form for this G_2 whose conjectural transfert to $\mathrm{GL}(7)$ is cuspidal should occur for the weight $k = 8$, which is the first integer k such that $G_2(2k, 2k - 2) \neq 0$. Modular forms for the anisotropic \mathbb{Q} -form of G_2 have also been studied by Gross, Lansky, Pollack and Savin in [GrS], [GrP], [LP] and [P], partly in order to find \mathbb{Q} -motives with Galois group of type G_2 , a problem initially raised by Serre. The automorphic forms they consider there are not of level 1, but of some prime level p and Steinberg at this prime p .

Observe that the last 5 orthogonal π of rank 7 given by Cor.1.9 (ii) are the first level 1 "truly orthogonal" algebraic π appearing in nature, in the sense that they are not deduced from some symplectic ones. Their Sato-Tate group is conjecturally the compact group $\mathrm{SO}(7, \mathbb{R})$ (see § 4.19).

Let us now give a small sample of results concerning $O(w_1, w_2, w_3, w_4)$ for $w_4 > 0$, see Table 9 and [ChR] for more values.

Corollary 1.11.** (i) $O(w_1, w_2, w_3, w_4)$ vanishes for $w_1 < 24$.

(ii) The (w_1, w_2, w_3, w_4) with $0 < w_4 < w_1 \leq 26$ such that $O(w_1, w_2, w_3, w_4) \neq 0$ are $(24, 18, 10, 4), (24, 20, 14, 2), (26, 18, 10, 2), (26, 18, 14, 6), (26, 20, 10, 4), (26, 20, 14, 8), (26, 22, 10, 6), (26, 22, 14, 2), (26, 24, 14, 4), (26, 24, 16, 2), (26, 24, 18, 8), (26, 24, 20, 6)$, and for all of them $O(26, v, u, t) = 1$.

The conjectural Sato-Tate groups of those 12 forms is $\mathrm{SO}(8, \mathbb{R})$. Let us mention here that we have in our database other numerical results concerning the non-connected \mathbb{Z} -group $\mathrm{O}(8)$. We hope to add them soon to the database [ChR].

We now discuss a bit more the methods and proofs. As already explained, a first important technical ingredient to obtain all the numbers above is to be able to compute, say given a finite subgroup Γ of a compact connected Lie group G , and given a finite dimensional irreducible representation V of G , the dimension

$$\dim V^\Gamma$$

of the subspace of vectors in V which are fixed by Γ . This general problem is studied in Chapter 2 (which is entirely unconditional). The main result there is an explicit general formula for $\dim V^\Gamma$ as a function of the extremal weight of V , which is made explicit in the cases alluded above. Our approach is to write

$$\dim V^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_V(\gamma)$$

where $\chi_V : G \rightarrow \mathbb{C}$ is the character of V . The formula we use for χ_V is a degeneration of the Weyl character formula which applies to possibly non regular elements and which was established in [ChCl]. Fix a maximal torus T in G , with character group X , and a set $\Phi^+ \subset X$ of positive roots for the root system (G, T) . Let V_λ be the irreducible representation with highest weight λ . Then

$$\dim V_\lambda^\Gamma = \sum_{j \in J} a_j e^{\frac{2i\pi}{N} \langle b_j, w_j(\lambda + \rho) - \rho \rangle} P_j(\lambda)$$

where N is the lcm of the orders of the elements of Γ , $a_j \in \mathbb{Q}(e^{2i\pi/N})$, b_j is a certain cocharacter of T , w_j is a certain element in the Weyl group W of (G, T) , and P_j is a certain rational polynomial on $X \otimes \mathbb{Q}$ which is a product of at most $|\Phi^+|$ linear forms. For each $\gamma \in T$ let $W_\gamma \subset W$ be the Weyl group of the connected centralizer of γ in G with respect to T . Then

$$|J| = \sum_{\gamma} |W/W_\gamma|$$

where γ runs over a set of representatives of the G -conjugacy classes of elements of Γ .

In practice, this formula for $\dim V_\lambda^\Gamma$ is quite insane. Consider for instance $G = \mathrm{SO}(7, \mathbb{R})$ and $\Gamma = W^+(E_7)$ the positive Weyl group of the root system of type E_7 : this is the case we need to compute $S(w_1, w_2, w_3)$. Then $|J| = 725$, $N = 2520$ and $|\Phi^+| = 9$: it is certainly impossible to explicitly write down this formula in the present paper. This is however nothing (in this case!) for a computer and we refer to § 2.4 for some details about the computer program we wrote using PARI/GP. Let us mention that we use in an important way some tables of Carter [Ca] giving the characteristic polynomials of all the conjugacy classes of a given Weyl group in its reflexion representation.

The second important ingredient we need is Arthur's multiplicity formula in a various number of cases. Concretely this amounts to determine a quite large collection of signs. This is discussed in details in Chapter 3, in which we specify Arthur's general results to the case of classical semisimple \mathbb{Z} -groups G . This leads first to a number of interesting properties of the automorphic representations π satisfying (i), (ii) and (iii) of this introduction. Of course, a special attention is given to the groups G with $G(\mathbb{R})$ compact, hence to the integral theory of quadratic forms. We restrict our study to the representations in the discrete spectrum of G which are unramified at each finite place. At the archimedean place we are led to review some properties of the packets of representations defined by Adams-Johnson in [AdJ]. We explain in particular in the appendix the parameterization of the elements of these packets by the characters of the dual component group in the spirit of Adams paper [Ad] in the discrete series case. For our purposes, we need to apply Arthur's results to a number of classical groups of small rank, namely

$$\mathrm{SL}(2), \mathrm{Sp}(4), \mathrm{Sp}(6), \mathrm{SO}(2, 2), \mathrm{SO}(3, 2), \mathrm{SO}(7), \mathrm{SO}(8) \text{ and } \mathrm{SO}(9).$$

When $G(\mathbb{R})$ is compact, Arthur's multiplicity formula takes a beautifully simple form, in which the half-sum of the positive roots on the dual side plays an important role. Let G be any semisimple \mathbb{Z} -group such that $G(\mathbb{R})$ is compact. We do not assume that G is classical here and state the general conjectural formula. Let

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}(2, \mathbb{C}) \rightarrow \widehat{G}$$

be a global Arthur parameter with finite centralizer and such that ψ_∞ is an Adams-Johnson parameter (see [A1], as well as §4.19 and the appendix). Denote by π_ψ the irreducible admissible representation of $G(\mathbb{A})$ which is $G(\widehat{\mathbb{Z}})$ -spherical and with the Satake parameters and infinitesimal character determined by ψ according to Arthur's recipe. Denote also by $e(\psi)$ the (finite) number of \widehat{G} -conjugacy classes of global Arthur parameters ψ' as above such that $\pi_{\psi'} \simeq \pi_\psi$. For most ψ we have $e(\psi) = 1$. The multiplicity $m(\pi_\psi)$ of π_ψ in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ should be given in general by

$$(3) \quad m(\pi_\psi) = \begin{cases} e(\psi) & \text{if } \rho_{|C_\psi}^\vee = \varepsilon_\psi, \\ 0 & \text{otherwise.} \end{cases}$$

As explained in the appendix, the group C_ψ is always an elementary abelian 2-group. The character ε_ψ is defined by Arthur in [A1]. The character ρ^\vee is defined as follows.

First, the centralizer in \widehat{G} of $\varphi_{\psi_\infty}(W_{\mathbb{C}})$ is a maximal torus \widehat{T} of \widehat{G} , so that $\varphi_{\psi_\infty}(z) = z^\lambda \bar{z}^{\lambda'}$ for some $\lambda \in \frac{1}{2}X_*(\widehat{T})$ and all $z \in W_{\mathbb{C}}$, and λ is dominant with respect to a unique Borel subgroup \widehat{B} of \widehat{G} containing \widehat{T} . Let ρ^\vee denote the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$. As G is semisimple over \mathbb{Z} and $G(\mathbb{R})$ is compact, this is actually a character of \widehat{T} . By construction, we have $C_\psi \subset \widehat{T}$, and thus formula (3) makes sense. The second important statement is that any automorphic representation of G which is $G(\widehat{\mathbb{Z}})$ -spherical has the form π_ψ for some ψ as above.

We end this introduction by discussing two other applications. The first one is very much in the spirit of the work of the first author and Lannes in [ChLa]. It concerns the genus of euclidean lattices $L \subset \mathbb{R}^{25}$ of covolume $\sqrt{2}$ which are *even*, in the sense that $x \cdot x \in 2\mathbb{Z}$ for each $x \in L$. A famous computation by Borchers in [Bo] asserts that there are up to isometry exactly 121 such lattices. It follows that there are exactly 121 automorphic representations of the reductive group $\mathrm{SO}(25)$ over \mathbb{Q} which is definite and split at all the finite places, which have conductor 1 and trivial coefficient. The dual group of $\mathrm{SO}(25)$ is $\mathrm{Sp}(24, \mathbb{C})$.

Observe now our tables : we have found exactly 22 cuspidal automorphic representations π of $\mathrm{GL}(n)$ (for any n) satisfying conditions (i), (ii) and (iii) above and with motivic weight ≤ 23 , namely :

- (a) The 7 representations of $\mathrm{PGL}(2)$ with Hodge number ≤ 23 ,
- (b) The 7 symplectic representations of $\mathrm{PGL}(4)$ with motivic weight ≤ 23 ,
- (c) The 7 symplectic representations of $\mathrm{PGL}(6)$ with motivic weight ≤ 23 ,
- (d) The orthogonal representation $\mathrm{Sym}^2 \Delta_{11}$ of $\mathrm{PGL}(3)$.

We have now this first crazy coincidence (see § 3.11 for the notion of global Arthur parameter), which is easy to check with a computer.

Proposition 1.12. *There are exactly 121 global Arthur parameters for $\mathrm{SO}(25)$ which have trivial infinitesimal character that one can form using only those 22 cuspidal automorphic representations.*

See the table of Chapter 12 for a list of these parameters. One uses the following notation : if $S(w_1, \dots, w_r) = 1$ we denote by Δ_{w_1, \dots, w_r} the unique $\pi \in \Pi(\mathrm{PGL}(2r))$ satisfying (i) to (iii) and with Hodge numbers w_1, \dots, w_r . When $S(w_1, \dots, w_r) = k > 1$ we denote by $\Delta_{w_1, \dots, w_r}^k$ any of the k representations of $\mathrm{PGL}(2r)$ with this latter properties.

The second miracle is that for each of these 121 parameters ψ , the unique level 1 automorphic representation of $\mathrm{SO}(25)(\mathbb{A})$ in the packet $\Pi(\psi)$ (see Def. 3.14) has indeed a nonzero multiplicity (i.e. multiplicity 1). In other words, we have the following theorem.

Theorem 1.13.** *The 121 level 1 automorphic representations of $\mathrm{SO}(25)$ with trivial coefficient are the ones given in Chapter 12.*

The 24 level 1 automorphic representations of $\mathrm{O}(24)$ with trivial coefficient ("associated" to the 24 Niemeier lattices) and the 32 level 1 automorphic representations of $\mathrm{SO}(23)$ with trivial coefficient (associated to the 32 even lattices of rank 23 of covolume

$\sqrt{2}$) had been determined in [ChLa]. As in [ChLa], observe that given the shape of Arthur's multiplicity formula, the naive probability that Theorem 1.13 be true was close to 0 (about 2^{-450} here in we take in account the size of C_ψ for each ψ), so something quite mysterious seems to occur for these small dimensions and trivial infinitesimal character. The miracle in all these cases is that whenever we can write down some ψ , then $\rho_{|C_\psi}^\vee$ is always equal to ε_ψ .

Proof — To check that each parameter has multiplicity one, we apply for instance the following simple claim already observed in [ChLa]. Let $\psi = (k, (n_i), (d_i), (\pi_i))$ be a global Arthur parameter for $\mathrm{SO}(8m \pm 1)$ with trivial infinitesimal character. Assume there exists an integer $1 \leq i \leq k$ such that $\pi_i = 1$ and π_j is symplectic if $j \neq i$. Then the unique $\pi \in \Pi(\psi)$ has a nonzero multiplicity (hence multiplicity 1) if and only if for each $j \neq i$ one has either $\varepsilon(\pi_j) = 1$ or $d_j < d_i$. Indeed, when the infinitesimal character of ψ is trivial the formula for $\rho^\vee(s_j)$ given in §3.23.1 shows that $\rho^\vee(s_j) = \varepsilon(\pi_j)$ for each $j \neq i$. The claim follows as $\varepsilon_\psi(s_j) = \varepsilon(\pi_j)^{\mathrm{Min}(d_i, d_j)}$ by definition (see §3.20), as d_i is even but d_j is odd.

Among the 21 symplectic π 's above of motivic weight ≤ 23 , one observes that exactly 4 of them have epsilon factor -1 , namely

$$\Delta_{17}, \Delta_{21}, \Delta_{23,9} \text{ and } \Delta_{23,13}.$$

A case-by-case check at the list concludes the proof thanks to this claim except for the parameter

$$\mathrm{Sym}^2 \Delta_{11}[2] \oplus \Delta_{11}[9]$$

which is the unique parameter which is not of the form above. But it is clear that for such a ψ one has $\varepsilon_\psi = 1$ and one observes that $\rho_{|C_\psi}^\vee = 1$ as well in this case, which concludes the proof (see §3.23.1). \square

None of the 121 automorphic representations above is tempered. This is clear since none of the 21 symplectic π 's above admit the Hodge number 1. Two representations in the list are not too far from being tempered however, namely

$$\Delta_{23,13,5} \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [4].$$

$$\Delta_{23,15,3} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [2].$$

It is thus tempting to consider the following problem : for which even integer n can we find a partition $n = \sum_{i=1}^r n_i$ in even integers n_i , as well as symplectic cuspidal automorphic representations π_i of $\mathrm{PGL}(n_i)$ satisfying the general assumptions (i), (ii), (iii), such that the induced representation

$$\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r$$

of $\mathrm{PGL}(n)(\mathbb{A})$ has the property that π_∞ has the same infinitesimal character as the trivial representation ? Observe that the L -function

$$L(\pi, s) = \prod_i L(\pi_i, s)$$

of such a π shares much of the analytic properties of the L -function of a cuspidal π' of $\mathrm{PGL}(n)$ satisfying (i) to (iii) and with Hodge numbers $1, 3, \dots, n-1$: they have the same archimedean factors and both satisfy Ramanujan's conjecture. In particular, it seems that the methods of [Fe], hence his results in §9 *loc. cit.*, apply to these more general L -functions. They say that such an L -function does not exist if $n < 23$, and even if $n = 24$ if one assumes the Riemann hypothesis. This is fortunately compatible with our previous result !

Our tables allow to show, on the other hand, that the above problem has a positive answer for $n = 28$, which shows the existence of a very interesting L -function in this dimension.

Theorem 1.14.** *There is a (non cuspidal, symplectic) automorphic representation of $\mathrm{PGL}(28)$ over \mathbb{Q} which satisfies (i), (ii) and (iii) and with the same infinitesimal character as the trivial representation, namely*

$$\Delta_{27,23,9,1} \oplus \Delta_{25,13,3} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}.$$

It simply follows from the observation that $S(27, 23, 9, 1) = S(25, 13, 3) = S(21, 5) = S(19, 7) = 1$. Actually, it is remarkable that our whole tables only allow to find a single representation with these properties. It seems reasonable to conjecture that this is indeed the only one and that there are no in rank $n = 26$. We have also considered the similar problem with orthogonal representations and found one (actually several) in dimension 31 (but no below).

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2. POLYNOMIAL INVARIANTS OF FINITE SUBGROUPS OF COMPACT CONNECTED LIE GROUPS

2.1. The setting. Let G be a compact connected Lie group and consider

$$\Gamma \subset G$$

a finite subgroup. Let V be a finite dimensional complex continuous representation of G . The general problem addressed in this chapter is to compute the dimension

$$\dim V^\Gamma$$

of the subspace $V^\Gamma = \{v \in V, \gamma(v) = v \ \forall \gamma \in \Gamma\}$ of Γ -invariants in V . Equivalently,

$$(4) \quad \dim V^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_V(\gamma)$$

where $\chi_V : G \rightarrow \mathbb{C}$ is the character of V . One may of course reduce to the case where V is irreducible and we shall most of the time do so. In order to apply formula (4) it is enough to know :

- (a) The value of the character χ_V on each conjugacy class in G ,
- (b) For each $\gamma \in \Gamma$, a representative of the conjugacy class $c(\gamma)$ of γ in G .

Of course, $c(\gamma)$ only depends on the conjugacy class of γ , but the induced map $c : \text{Conj}(\Gamma) \rightarrow \text{Conj}(G)$ needs not to be injective in general. Here $\text{Conj}(H)$ denotes the set of conjugacy classes of the group H .

We will be especially interested in cases where $\Gamma \subset G$ are fixed, but with V varying over all the possible irreducible representations of G . With this in mind, observe that problem (b) has to be solved once, but problem (a) for infinitely many V whenever $G \neq \{1\}$.

Consider for instance the group $\Gamma \subset \text{SO}(3, \mathbb{R})$ of positive isometries of a given regular tetrahedron in the euclidean \mathbb{R}^3 with center 0. Each numbering of the vertices of the tetrahedron defines an isomorphism

$$\Gamma \simeq \mathfrak{A}_4$$

and we fix one. For each odd integer $n \geq 1$ denote by V_n the n -dimensional irreducible representations of $\text{SO}(3, \mathbb{R})$. This representation V_n is well-known to be unique up to isomorphism, and if $g_\theta \in \text{SO}(3, \mathbb{R})$ is a non-trivial rotation with angle θ then

$$\chi_{V_n}(g_\theta) = \frac{\sin(n\frac{\theta}{2})}{\sin(\frac{\theta}{2})}.$$

The group Γ has 4 conjugacy classes, with representatives $1, (12)(34), (123), (132)$ and respective orders $1, 3, 4, 4$. These representatives act on \mathbb{R}^3 as rotations with respective

angles $0, \pi, 2\pi/3, 2\pi/3$. Observe that (123) and (132) are conjugate in $\mathrm{SO}(3, \mathbb{R})$ but not in Γ . Formula (4) thus writes

$$\dim V_n^\Gamma = \frac{1}{12} \left(n + 3 \frac{\sin(n\pi/2)}{\sin(\pi/2)} + 8 \frac{\sin(n\pi/3)}{\sin(\pi/3)} \right) = \begin{cases} \lceil \frac{n}{12} \rceil & \text{if } n \equiv 1, 7, 9 \pmod{12}, \\ \lfloor \frac{n}{12} \rfloor & \text{if } n \equiv 3, 5, 11 \pmod{12}. \end{cases}$$

This formula is quite simple but already possesses some features of the general case.

2.2. The degenerate Weyl character formula. A fundamental ingredient for the above approach is a formula for the character $\chi_V(g)$ where V is any irreducible representations of G and $g \in G$ is any element as well. When g is either central or regular, such a formula is given by Weyl's dimension formula and Weyl's character formula respectively. These formulas have been extended by Kostant to the more general case where the centralizer of g is a Levi subgroup of G , and by the first author and Clozel in general in [ChCl, Prop. 1.9]. Let us now recall this last result.

We fix once and for all a maximal torus $T \subset G$ and denote by

$$X = X^*(T) = \mathrm{Hom}(T, \mathbb{S}^1)$$

the character group of T . We denote by $\Phi = \Phi(G, T) \subset X \otimes \mathbb{R}$ the root system of (G, T) and $W = W(G, T)$ its Weyl group. We choose $\Phi^+ \subset \Phi$ a system of positive roots, say with base Δ , and we fix as well a W -invariant scalar product (\cdot, \cdot) on $X \otimes \mathbb{R}$. Recall that a dominant weight is an element $\lambda \in X$ such that $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta$. The Cartan-Weyl theory defines a canonical bijection

$$\lambda \mapsto V_\lambda$$

between the dominant weights and the irreducible representations of G . The representation V_λ is uniquely characterized by the following property. If V is a representation of G , denote by $P(V) \subset X$ the subset of $\mu \in X$ appearing in $V|_T$. If we consider the partial ordering on X defined by $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda$ is a finite sum of elements of Δ , then λ is the maximal element of $P(V_\lambda)$. One says that λ is the highest weight of V_λ .

Let us fix some dominant weight $\lambda \in X$. Recall that the inclusion $T \subset G$ induces a bijection

$$W \backslash T \xrightarrow{\sim} \mathrm{Conj}(G),$$

it is thus enough to determine $\chi_{V_\lambda}(t)$ for any $t \in T$. Fix some $t \in T$ and denote by

$$M = C_G(t)^0$$

the neutral component of the centralizer of t in G . Of course, $t \in T \subset M$ and T is maximal torus of M . Set $\Phi_M^+ = \Phi(M, T) \cap \Phi^+$ and consider the set

$$W^M = \{w \in W, w^{-1}\Phi_M^+ \subset \Phi^+\}.$$

Let ρ and $\rho_M \in X \otimes \mathbb{R}$ denote respectively the half-sum of the elements of Φ^+ and of Φ_M^+ . If $w \in W^M$, we set $\lambda_w = w(\lambda + \rho) - \rho_M \in X \otimes \mathbb{R}$. Observe that

$$2 \frac{(\alpha, \lambda_w)}{(\alpha, \alpha)} \in \mathbb{N}, \quad \forall \alpha \in \Phi_M^+.$$

It follows that λ_w is a dominant weight for some finite covering of M , that we may choose to be the smallest finite covering $\widetilde{M} \rightarrow M$ for which $\rho - \rho_M$ becomes a character. This is possible as $2\frac{(\alpha, \rho - \rho_M)}{(\alpha, \alpha)} \in \mathbb{Z}$, $\forall \alpha \in \Phi_M^+$. It follows from the Weyl dimension formula that the dimension of the irreducible representation of \widetilde{M} with highest weight λ_w is $P_M(\lambda_w)$ where we set

$$P_M(v) = \prod_{\alpha \in \Phi_M^+} \frac{(\alpha, v + \rho_M)}{(\alpha, \rho_M)} \quad \forall v \in X \otimes \mathbb{R}.$$

We need two last notations before stating the main result. We denote by $\varepsilon : W \rightarrow \{\pm 1\}$ the signature, and for $x \in X$ it will be convenient to write t^x for $x(t)$. It is well-known that $w(\mu + \rho) - \rho \in X$ for all $w \in W$ and $\mu \in X$.

Proposition 2.3. (*Degenerate Weyl character formula*) *Let $\lambda \in X$ be a dominant weight, $t \in T$ and $M = C_G(t)^0$. Then*

$$\chi_{V_\lambda}(t) = \frac{\sum_{w \in W^M} \varepsilon(w) \cdot t^{w(\lambda + \rho) - \rho} \cdot P_M(w(\lambda + \rho) - \rho_M)}{\prod_{\alpha \in \Phi^+ \setminus \Phi_M^+} (1 - t^{-\alpha})}.$$

Proof — This is the last formula in the proof of [ChCl, Prop. 1.9]. Note that it is unfortunately incorrectly stated in the beginning of that proof that up to replacing G by a finite covering one may assume that ρ and ρ_M are characters. It is however not necessary for the proof to make any reduction on the group G . Indeed, we rather have to introduce the inverse image \widetilde{T} of T in the covering \widetilde{M} defined above and argue as *loc. cit.* but in the Grothendieck group of characters of \widetilde{T} . The argument given there shows that for any element $z \in \widetilde{T}$ whose image in T is t , we have

$$\chi_{V_\lambda}(t) = z^{\rho_M - \rho} \frac{\sum_{w \in W^M} \varepsilon(w) z^{\lambda_w} P_M(\lambda_w)}{\prod_{\alpha \in \Phi^+ \setminus \Phi_M^+} (1 - t^{-\alpha})}.$$

We conclude as $\lambda_w + \rho_M - \rho = w(\lambda + \rho) - \rho \in X$, so $z^{\rho_M - \rho} z^{\lambda_w} = t^{w(\lambda + \rho) - \rho}$. \square

2.4. A computer program. We now return to the main problem discussed in §2.1. We fix a compact connected Lie group G and a finite subgroup $\Gamma \subset G$. In order to enumerate the irreducible representations of G we fix as in the previous paragraph a maximal torus $T \subset G$ and a subset Φ^+ of positive roots for (G, T) . For each dominant weight λ one thus has a unique irreducible representation V_λ with highest weight λ , hence a number

$$\dim(V_\lambda^\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_{V_\lambda}(t(\gamma)),$$

where for each $\gamma \in \Gamma$ we define $t(\gamma)$ to be any element in T which is conjugate to γ in G . The last ingredient to be given for the computation is thus a list of these elements $t(\gamma) \in T$, which is a slightly more precise form of problem (b) of §2.1. Recall that the elements of T may be described as follows. Denote by $X^\vee = \text{Hom}(\mathbb{S}^1, T)$ the cocharacter

group of T and $\langle, \rangle : X \otimes X^\vee \rightarrow \mathbb{Z}$ the canonical perfect pairing. If $\mu \in X^\vee \otimes \mathbb{C}$, denote by $e^{2i\pi\mu}$ the unique element $t \in T$ such that

$$\forall \lambda \in X, \quad \lambda(t) = e^{2i\pi\langle \lambda, \mu \rangle}.$$

The map $\mu \mapsto e^{2i\pi\mu}$ defines an isomorphism $(X^\vee \otimes \mathbb{C})/X^\vee \xrightarrow{\sim} T$.

We thus wrote a computer program with the following property. It takes as input :

- (a) The based root datum of (G, T, Φ^+) , i.e. the collection $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \langle, \rangle, \iota)$, where $\Phi^\vee \subset X^\vee$ is the set of coroots of (G, T) and $\iota : \Phi \rightarrow \Phi^\vee$ is the bijection $\alpha \mapsto \alpha^\vee$.
- (b) A finite set of pairs $(\mu_j, C_j)_{j \in J}$, where $\mu_j \in X^\vee \otimes \mathbb{Q}$ and $n_j \in \mathbb{N}$, with the property that there exists a partition $\Gamma = \coprod_{j \in J} \Gamma_j$ such that $|\Gamma_j| = C_j$ and each element $\gamma \in \Gamma_j$ is conjugate in G to the element $e^{2i\pi\mu_j} \in T$.
- (c) A dominant weight $\lambda \in X$.

It returns $\dim(V_\lambda^\Gamma) = |\Gamma|^{-1} \sum_{j \in J} C_j \chi_{V_\lambda}(e^{2i\pi\mu_j})$.

Recall that for $\alpha \in \Phi^+$ and $v \in X \otimes \mathbb{R}$ one has the relation $2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \langle v, \alpha^\vee \rangle$, thus (a), (b) and (c) contain indeed everything needed to evaluate the degenerate Weyl character formula. Although in theory the Weyl group W of (G, T) may be deduced from (a) we also take it as an input in practice. The program computes in particular for each $t_j = e^{2i\pi\mu_j}$ the root system of $M_j = C_G(t_j)^0$ and the set W^{M_i} . Of course it is often convenient to take $X = X^\vee = \mathbb{Z}^n$ with the canonical pairing. A routine in PARI/GP may be found at the url [ChR].

2.5. Some numerical applications. We shall present in this paper four numerical applications of our computations. They concern the respective compact groups

$$G = \mathrm{SO}(7, \mathbb{R}), \quad \mathrm{SO}(8, \mathbb{R}), \quad \mathrm{SO}(9, \mathbb{R}), \quad \text{and } G_2$$

and each time a very specific finite subgroup Γ . We postpone to § 8.2 the discussion of the case G_2 and concentrate here on the first three cases. The general context is as follows.

Let V be a finite dimensional vector space over \mathbb{R} and let $R \subset V$ be a reduced root system in the sense of Bourbaki [Bki, Chap. VI §1]. Let $W(R)$ denote the Weyl group of R and fix a $W(R)$ -invariant scalar product on V , so that

$$W(R) \subset \mathrm{O}(V).$$

Assume that R is irreducible. Then V is irreducible as a representation of $W(R)$ ([Bki, Chap. VI §2]). Let $\varepsilon : W(R) \rightarrow \{\pm 1\}$ the signature of $W(R)$, i.e. $\varepsilon(w) = \det(w)$ for each $w \in W(R)$, and set

$$W(R)^+ = W(R) \cap \mathrm{SO}(V).$$

We are in the general situation of this chapter with $G = \mathrm{SO}(V)$ and $\Gamma = W(R)^+$. Beware that the root system Φ of $(\mathrm{SO}(V), T)$ is not the root system R above ! We choose the standard based root datum for $(\mathrm{SO}(V), T)$ as follows. If $l = \lfloor \frac{\dim(V)}{2} \rfloor$ we set

$X = X^\vee = \mathbb{Z}^l$, equipped with the canonical pairing : if (e_i) denotes the canonical basis of \mathbb{Z}^n , then $\langle e_i, e_j \rangle = \delta_{i=j}$. There are two cases depending whether $\dim(V)$ is odd or even :

- (i) $\dim(V) = 2l + 1$. Then $\Phi^+ = \{e_i, e_i \pm e_j, 1 \leq i < j \leq n\}$, $e_i^\vee = 2e_i$ for all i , and $(e_i \pm e_j)^\vee = e_i \pm e_j$ for all $i < j$.
- (ii) $\dim(V) = 2l$. Then $\Phi^+ = \{e_i \pm e_j, 1 \leq i < j \leq n\}$ and $(e_i \pm e_j)^\vee = e_i \pm e_j$ for all $i < j$.

The dominant weights are thus the $\lambda = (n_1, \dots, n_l) = \sum_{i=1}^l n_i e_i \in X$ such that $n_1 \geq n_2 \geq \dots \geq n_l \geq 0$ if $\dim(V) = 2l + 1$, and such that $n_1 \geq n_2 \geq \dots \geq n_{l-1} \geq |n_l|$ if $\dim(V) = 2l$.

Consider now the input (b) for the program. Recall that at least if $\dim(V)$ is odd, the conjugacy class of any element $g \in \mathrm{SO}(V)$ is uniquely determined by the characteristic polynomial of g acting on V . It turns out that for any reduced root system R , the characteristic polynomial of each conjugacy class of elements of $W(R)$ has been determined by Carter in [Ca]. We make an important use of these results, especially when R is of type E_7 and E_8 for the applications here, in which case it is given in Tables 10 and 11 *loc. cit.*

2.5.1. Case I : R is of type E_7 . Then $-1 \in W(R)$ and $W(R) = W(R)^+ \times \{\pm 1\}$, so the conjugacy classes in $W(R)^+$ coincide with the conjugacy classes in $W(R)$ belonging to $W(R)^+$, i.e. with determinant 1. From Table 10 *loc. cit.* one sees that $W(R)^+$ has exactly 27 conjugacy classes (c_j) and for each of them it gives its order C_j and its characteristic polynomial, from which we deduce μ_j : this is the datum we need for (b). For each dominant weight $\lambda = (n_1, n_2, n_3) \in \mathbb{Z}^3$ our computer program then returns $\dim(V_\lambda)^{W(R)^+}$: see Table 2 for a sample of results and to the url [ChR] for much more.

2.5.2. Case II : R is of type E_8 . This case presents two little differences compared to the previous one. First the characteristic polynomial of an element $g \in \mathrm{SO}(V)$ does only determine its $\mathrm{O}(V)$ -conjugacy class as $\dim(V) = 8$ is even. It determine its $\mathrm{SO}(V)$ conjugacy class if and only if ± 1 is not an eigenvalue of g . Let $C \subset W(R)^+$ be a $W(R)$ -conjugacy class and let P be its characteristic polynomial. If ± 1 is a root of P , there is thus a unique conjugacy class in $\mathrm{SO}(V)$ with this characteristic polynomial. Otherwise, C meets exactly two conjugacy class in $\mathrm{SO}(V)$, it follows that $C = C_1 \amalg C_2$ where the C_i are $W(R)^+$ -conjugacy classes permuted by any element in $W(R) \setminus W(R)^+$, and in particular $|C_1| = |C_2|$. It follows that the table of Carter gives input (b) as well in this case.

We refer to Table 3 for a sample of values of nonzero $\dim(V_\lambda^{W(R)^+})$ for $\lambda = (n_1, n_2, n_3, n_4)$ dominant with⁷ $n_4 \geq 0$. As $-1 \in W(R)^+$, one must have $n_1 + n_2 + n_3 + n_4 \equiv 0 \pmod{2}$.

⁷One easily sees that $\dim V_\lambda^{W(R)^+} = \dim V_{\lambda'}^{W(R)^+}$ if $\lambda = (n_1, n_2, n_3, n_4)$ and $\lambda' = (n_1, n_2, n_3, -n_4)$. Better, the triality $(n_1, n_2, n_3, n_4) \mapsto (\frac{n_1+n_2+n_3+n_4}{2}, \frac{n_1+n_2-n_3-n_4}{2}, \frac{n_1-n_2+n_3-n_4}{2}, \frac{-n_1+n_2+n_3-n_4}{2})$ preserves as well the table. This has a natural explanation when we identify $W(R)$ as a certain orthogonal group over \mathbb{Z} as in § 7.1, see [Gr1].

2.5.3. *Case III : the Weyl group of E_8 as a subgroup of $\mathrm{SO}(9, \mathbb{R})$.* This case is slightly different and we start with some general facts, keeping the setting of the beginning of § 2.5. Consider now the representation of $W(R)$ on $V \oplus \mathbb{R}$ defined by $V' = V \oplus \varepsilon$. The map $w \mapsto (w, \varepsilon(w))$ defines an injective group homomorphism

$$W \hookrightarrow \mathrm{SO}(V'),$$

and we are thus again in the general situation of this chapter with this time $G = \mathrm{SO}(V')$ and $\Gamma = W(R)$.

Consider now the special case of a R of type E_8 , so that $\dim(V') = 9$. Table 11 of Carter gives the characteristic polynomials for the action of V of each $W(R)$ -conjugacy class in $W(R)$, from which we immediately deduce the characteristic polynomial for the action of $V' = V \oplus \varepsilon$, hence the associated conjugacy class in $\mathrm{SO}(V')$ as $\dim(V') = 9$ is odd. This is the datum (b) we need for computing $\dim(V_\lambda)^{W(R)^+}$: see Table 4 for a sample of values.

2.6. Reliability. Of course, there is some possibility that we have made mistakes during the implementation of the program of § 2.4 or of the characteristic polynomials from Carter's tables. This seems however unlikely due to the very large number of verifications we have made.

The first trivial check is that the sum of characteristic polynomials of all the elements of Γ in cases I and II is

$$|W(R)^+|(X^{\dim(V)} + (-1)^{\dim(V)})$$

as it should be. Indeed, V is an irreducible representation of $W(R)$, and even of $W(R)^+$ in both cases.

The second check is that our computer program for $\dim(V_\lambda^\Gamma)$ always returns a positive integer ... and it does in the several hundreds of cases we have tried. As observed in the introduction, a priori each term in the sum of the degenerate Weyl character formula is not an integer but an element of the cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is a N -th root of unity ($N = 2520$ in both cases, and we indeed computed in this number field with PARI GP). This actually makes a really good check for both the degenerate Weyl character formula and Carter's tables.

We will present two more evidences in the paper. One just below using a specific family of irreducible representations of $W(R)^+$ for which one can compute directly the dimension of the $W(R)^+$ -invariants. The other one will be done much later in Chapters 5, 6, 7, where we shall check that our computations beautifully confirm the quite intricate Arthur's multiplicity formula in a large number of cases as well.

2.7. A check : the harmonic polynomial invariants of a Weyl group. We keep the notations of § 2.5. For each integer $n \geq 0$, let $\mathrm{Pol}_n(V)$ denote the space homogeneous polynomials on V of degree n and consider the two formal power series in $\mathbb{Z}[[t]]$:

$$P_R(t) = \sum_{n \geq 0} \dim(\mathrm{Pol}_n(V)^{W(R)}) t^n,$$

$$A_R(t) = \sum_{n \geq 0} \dim((\text{Pol}_n(V) \otimes \varepsilon)^{W(R)}) t^n.$$

By [Bki, Chap. V §6], if $l = \dim(V)$ and m_1, \dots, m_l are the exponents of $W(R)$, then

$$P_R(t) = \prod_{i=1}^l (1 - t^{m_i+1})^{-1} \quad \text{and} \quad A_R(t) = t^{|R|/2} P_R(t).$$

Let Δ be "the" $O(V)$ -invariant Laplace operator on V . It induces an $O(V)$ -equivariant surjective morphism $\text{Pol}_{n+2}(V) \rightarrow \text{Pol}_n(V)$, whose kernel

$$H_n(V) \subset \text{Pol}_n(V)$$

is the space of harmonic polynomials of degree n on V . This is an irreducible representation of $SO(V)$ if $\dim(V) \neq 2$, namely the irreducible representation with highest weight $ne_1 = (n, 0, \dots, 0)$ (see e.g. [GW, §5.2.3]). One deduces the following corollary.

Corollary 2.8. (i) $\sum_{n \geq 0} \dim(H_n(V)^{W(R)^+}) t^n = (1 - t^2)(1 + t^{|R|/2}) P_R(t)$.
(ii) $\sum_{n \geq 0} \dim(H_n(V')^{W(R)}) t^n = (1 + t^{|R|/2}) P_R(t)$.

We are not aware of an infinite family (V_i) of irreducible representations of $SO(V)$ other than the $H_i(V)$ with a simple close formula for $\dim V_i^{W(R)^+}$.

We end with some examples. Consider for instance the special case where R has type E_7 . The exponents of R are 1, 5, 7, 9, 11, 13, 17, and $|R| = 18 \cdot 7 = 126$. The power series of the corollary (i) thus becomes

$$\begin{aligned} & \frac{1 + t^{63}}{(1 - t^6)(1 - t^8)(1 - t^{10})(1 - t^{12})(1 - t^{14})(1 - t^{18})} \\ &= 1 + t^6 + t^8 + t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + 4t^{18} + 4t^{20} + 4t^{22} + 7t^{24} + 7t^{26} + 8t^{28} + o(t^{28}) \end{aligned}$$

The Case $R = E_8$ is similar, with exponents 1, 7, 11, 13, 17, 19, 23, 29 and $|R| = 8 \cdot 30 = 240$, the power series of the corollary (i)

$$\begin{aligned} & \frac{1 + t^{120}}{(1 - t^8)(1 - t^{12})(1 - t^{14})(1 - t^{18})(1 - t^{20})(1 - t^{24})(1 - t^{30})} \\ &= 1 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + 2t^{20} + t^{22} + 3t^{24} + 2t^{26} + 3t^{28} + 3t^{30} + 5t^{32} + 3t^{34} + 6t^{36} + o(t^{36}) \end{aligned}$$

The power series in case (ii) for R of type E_8 is the one above multiplied by $(1 - t^2)^{-1}$, which starts with

$$1 + t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + 3t^{12} + 4t^{14} + 5t^{16} + 6t^{18} + 8t^{20} + 9t^{22} + 12t^{24} + 14t^{26} + 17t^{28} + o(t^{28})$$

In the three cases, those numbers turn out to perfectly fit our computations of the previous paragraph with the degenerate Weyl character formula!

3. AUTOMORPHIC REPRESENTATIONS OF CLASSICAL GROUPS : REVIEW OF ARTHUR'S RESULTS

Let us review Arthur's recent results on the endoscopic classification of discrete automorphic representations of classical groups. Our main aim is to apply it to certain classical groups which are reductive groups over \mathbb{Z} , for which the theory is simpler.

3.1. Classical semisimple groups over \mathbb{Z} . Besides the \mathbb{Z} -groups $\mathrm{SL}(n)$, the symplectic groups $\mathrm{Sp}(2g)$ and their isogeny classes, we shall mainly focus on the following collection of orthogonal groups.

A *quadratic form over \mathbb{Z}* will be the datum of a free abelian group L of finite rank, say n , equipped with a quadratic form $q : L \rightarrow \mathbb{Z}$ such that the associated symmetric \mathbb{Z} -bilinear form $x \cdot y = q(x + y) - q(x) - q(y)$ has determinant ± 1 if n is even, ± 2 if n is odd. Note that $x \cdot x = 2q(x) \in 2\mathbb{Z}$, thus $(x, y) \mapsto x \cdot y$ is alternate modulo 2, which forces n to be even if $x \cdot y$ is a perfect pairing. If L is a quadratic form over \mathbb{Z} , the special orthogonal group scheme of L over \mathbb{Z}

$$\mathrm{SO}_L$$

is then reductive over \mathbb{Z} , even semisimple if $n \neq 2$, see [Gr1] or [ChLa]. We shall also denote by $\mathrm{O}(L)$ the orthogonal group of L and by $\mathrm{SO}(L)$ its subgroup whose elements have determinant 1.

Let L be a quadratic form over \mathbb{Z} of rank n . We denote by (p, q) the signature of $L \otimes_{\mathbb{Z}} \mathbb{R}$. As is well known [Se1], we have $p - q \equiv -1, 0, 1 \pmod{8}$, and conversely any (p, q) with $p, q \geq 0$ satisfying this condition is the signature of a quadratic form L over \mathbb{Z} of rank $p + q$. When $pq \neq 0$, such an L is even unique up to isomorphism, and we shall simply denote SO_L by

$$\mathrm{SO}(p, q).$$

As an example, have the following exceptional but useful isomorphisms over \mathbb{Z} :

$$\mathrm{SO}(1, 1) = \mathbb{G}_m, \quad \mathrm{SO}(2, 1) = \mathrm{PGL}(2), \quad \mathrm{SO}(3, 2) = \mathrm{PGSp}(4),$$

as well as a central isogeny $\mathrm{SO}(2, 2) \rightarrow \mathrm{PGL}(2) \times \mathrm{PGL}(2)$.

If L is a quadratic form over \mathbb{Z} of rank n , then $L \otimes \mathbb{Z}_p$ contains as a direct factor the hyperbolic quadratic form over $\mathbb{Z}_p^{[n/2]}$ for each prime p ; thus $L \otimes \mathbb{Z}_p$ is hyperbolic for n even). In standard terminology, they form thus a single genus for each signature. By the theory of Hasse-Minkowski, one checks that the rational quadratic form $v \mapsto v \cdot v$ on $L \otimes \mathbb{Q}$ is isomorphic to the standard quadratic form $\langle 1, \dots, 1, -1, \dots, -1 \rangle$ if n is even, and to $\langle 1, \dots, 1, -1, \dots, -1 \rangle \oplus \langle \pm 2 \rangle$ if n is odd.

We shall actually be mostly interested in the definite case $pq = 0$, for which the situation is quite different, and which only exists in rank $n \equiv -1, 0, 1 \pmod{8}$ as we said. There is no loss of generality in restricting to the positive ones. Consider the standard euclidean space \mathbb{R}^n , equipped with the scalar product $(x_i) \cdot (y_i) = \sum_i x_i y_i$ and denote by

$$\mathcal{L}_n$$

the set of lattices $L \subset \mathbb{R}^n$ such that the map $x \mapsto \frac{x \cdot x}{2}$ defines an integral quadratic form on L . It is equivalent to ask that L is even, i.e. $x \cdot x \in 2\mathbb{Z}$ for each $x \in L$, L has covolume 1 if n is even, and L has covolume $\sqrt{2}$ otherwise. The action of $O(n, \mathbb{R})$ on \mathbb{R}^n induces an action as well on \mathcal{L}_n and we shall denote by

$$X_n = O(n, \mathbb{R}) \backslash \mathcal{L}_n$$

the quotient set. The map $L \subset L \otimes \mathbb{R}$ defines a bijection between the set of isomorphism classes of quadratic forms over \mathbb{Z} of rank n and X_n . The set X_n is a finite set by reduction theory. Here is what seems to be currently known about $h_n = |X_n|$:

$$h_1 = h_7 = h_8 = h_9 = 1, \quad h_{15} = h_{16} = 2, \quad h_{17} = 4, \quad h_{23} = 32, \quad h_{24} = 24, \quad h_{25} = 121.$$

(Mordell, Witt, Kneser, Niemeier, Borchers) In all those cases explicit representatives of X_n are known : see [Bo], [CS]. When $n \geq 31$ then the Minkowski-Siegel-Smith mass formula shows that X_n is huge, and h_n has not been determined in any case. One sometimes need to introduce the set

$$\tilde{X}_n = SO(n, \mathbb{R}) \backslash \mathcal{L}_n$$

of $SO(n, \mathbb{R})$ -isometry classes of even lattices $L \subset \mathbb{R}^n$ as above. One has then a natural surjective map $\tilde{X}_n \rightarrow X_n$. The inverse image of the class of the lattice L has one element if $O(L) \neq SO(L)$, and two elements otherwise. In particular, $\tilde{X}_n = X_n$ if n is odd.

Some examples of such lattices are related to root systems as follows. If $R \subset \mathbb{R}^n$ is a root system of rank n such that each $x \in R$ satisfies $x \cdot x = 2$, let us denote by $Q(R)$, or even simply by R , the \mathbb{Z} -lattice it generates equipped with the euclidean quadratic form $\frac{1}{2} \sum_i x_i^2$. The Cartan matrix of the root system R is then symmetric and is a Gram matrix for the bilinear form of $Q(R)$. Its determinant is the index of connexion of R . It follows that A_1 , E_7 and E_8 are quadratic forms over \mathbb{Z} in respective ranks $n = 1, 7$ and 8 . As recalled above, they are even the unique such lattices in these dimensions. In general dimension $n = 8k + s$ with $s = -1, 0, 1 \bmod 8$, we get examples by choosing $E_7 \oplus E_8^{k-1}$, E_8^k and $E_8^k \oplus A_1$. We shall call them the *standard* positive quadratic forms over \mathbb{Z} and simply denote by

$$SO(n)$$

the special orthogonal group scheme of this lattice. Last but not least, the quadratic form over \mathbb{Z} of signature (p, q) with $p > q$ is the direct sum of the hyperbolic quadratic form of rank $2q$ and of any element of X_{p-q} .

Remark 3.2. Let L be the standard quadratic form of even rank n and signature (p, q) with $p \geq q$. Observe that L always contains elements α such that $\alpha \cdot \alpha = 2$. In particular, if s_α denotes the orthogonal symmetry with respect to such an α then $s_\alpha \in O(L)$ and it defines a section of the morphism $O_L \rightarrow \mathbb{Z}/2\mathbb{Z}$.

3.3. Discrete automorphic representations. If G is any semisimple group over \mathbb{Z} we denote by $\Pi(G)$ the set of $\pi = \pi_\infty \otimes \pi_f$, where π_f is a smooth irreducible complex representation of $G(\mathbb{A}_f)$ such that $\pi^{G(\widehat{\mathbb{Z}})} \neq 0$, and where π_∞ is an irreducible unitary representation of $G(\mathbb{R})$. Of course here $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ is the adèle ring of \mathbb{Q} . Denote

by $\mathcal{H}(G)$ the complex Hecke-algebra of the pair $(G(\mathbb{A}_f), G(\widehat{\mathbb{Z}}))$. A well-known result of Satake and Tits ensures that $\mathcal{H}(G)$ is commutative, so that $\dim \pi_f^{G(\widehat{\mathbb{Z}})} = 1$ for π_f as above.

Recall that the homogeneous space $G(\mathbb{Q}) \backslash G(\mathbb{A})$ has a $G(\mathbb{A})$ -invariant Radon measure (Weil) of finite volume (Borel, Harish-Chandra). Consider the Hilbert space

$$\mathcal{L}(G) = L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}}))$$

of square-integrable functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ for this measure which are $G(\widehat{\mathbb{Z}})$ -invariant on the right. This space $\mathcal{L}(G)$ is equipped with a unitary representations of $G(\mathbb{R})$ by right translations and of a commuting action of the Hecke algebra $\mathcal{H}(G)$. The subspace $\mathcal{L}_{\text{disc}}(G) \subset \mathcal{L}(G)$ is defined as the closure of the direct sum of the irreducible closed submodule for the $G(\mathbb{R})$ -action, it is stable by $\mathcal{H}(G)$. A fundamental result of Harish-Chandra asserts that each such module occurs with finite multiplicity. It follows that

$$(5) \quad \mathcal{L}_{\text{disc}}(G) = \overline{\bigoplus_{\pi \in \Pi(G)} m(\pi) \pi_{\infty} \otimes \pi_f^{G(\widehat{\mathbb{Z}})}},$$

where $m(\pi) \in \mathbb{N}$ is the multiplicity of π as a sub-representation of $\mathcal{L}(G)$. We denote by

$$\Pi_{\text{disc}}(G) \subset \Pi(G)$$

the subset of π such that $m(\pi) \neq 0$ and call them the *discrete automorphic representations of the \mathbb{Z} -group G* . A classical result of Gelfand and Piatetski-Shapiro asserts that the subspace of cuspforms of G , which is stable by $G(\mathbb{R})$ and $\mathcal{H}(G)$, is included in $\mathcal{L}_{\text{disc}}(G)$ and we denote by

$$\Pi_{\text{cusp}}(G) \subset \Pi_{\text{disc}}(G)$$

the subset of π consisting of cusp forms.

3.4. The case of Chevalley and definite semisimple \mathbb{Z} -group. All those automorphic representations have various models according to what G is and the type of π_{∞} . We shall content ourselves with the following classical descriptions. Consider the class set of G

$$\text{Cl}(G) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / G(\widehat{\mathbb{Z}}).$$

This is a finite set (Borel) and we set $h(G) = |\text{Cl}(G)|$. A well-known elementary fact is that

$$h(\text{SL}(n)) = h(\text{PGL}(n)) = h(\text{Sp}(n)) = h(\text{PGSp}(n)) = 1,$$

and what we said in § 3.1 about indefinite quadratic forms over \mathbb{Z} amounts to say as well that $h(\text{SO}(p, q)) = 1$ if $pq \neq 0$. More generally, the strong approximation theorem ensures that $h(G) = 1$ if G is simply connected and $G(\mathbb{R})$ has no compact factor (Kneser). Recall that a *Chevalley group* is a split semisimple \mathbb{Z} -group. We refer to [SGA3] and [Co] for the general theory of Chevalley groups.

Proposition 3.5. *Let G be a Chevalley group. Then $h(G) = 1$ and the inclusion $G(\mathbb{R}) \rightarrow G(\mathbb{A})$ induces a homeomorphism*

$$G(\mathbb{Z}) \backslash G(\mathbb{R}) \rightarrow G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}}).$$

Moreover, $G(\mathbb{R})/G(\mathbb{Z})$ is connected and $Z(G)(\mathbb{R}) = Z(G)(\mathbb{Z})$.

Proof — We refer to [SGA3, Exp. XXII §4.2, §4.3] and [Co, Chap. 6] for central isogenies between semisimple group schemes. Let $s : G_{\text{sc}} \rightarrow G$ be the central isogeny with G_{sc} simply connected. The \mathbb{Z} -group G_{sc} is a Chevalley group as well. Let T_{sc} be a maximal \mathbb{Z} -split torus in G_{sc} and set $T = s(T_{\text{sc}})$. Then $T_{\text{sc}} = G_{\text{sc}} \times_G T$ and if $\iota : T \rightarrow G$ and $\iota_{\text{sc}} : T_{\text{sc}} \rightarrow G_{\text{sc}}$ denote the inclusions we have an fppf-exact sequence on $\text{Spec}(\mathbb{Z})$

$$1 \longrightarrow T_{\text{sc}} \xrightarrow{\iota_{\text{sc}} \times 1/s} G_{\text{sc}} \times T \xrightarrow{s \times \iota} G \longrightarrow 1.$$

As T_{sc} is \mathbb{Z} -split, it follows from Hilbert 90 that this exact sequence remains exact on A -points for any commutative ring A . In particular, $G(\mathbb{A}_f) = s(G_{\text{sc}}(\mathbb{A}_f))T(\mathbb{A}_f)$. But $T(\mathbb{A}_f) = T(\mathbb{Q})(T(\mathbb{R}) \times T(\widehat{\mathbb{Z}}))$ as T is \mathbb{Z} -split as well. One deduces now $h(G) = 1$ from $h(G_{\text{sc}}) = 1$. The map of the first statement is trivially injective, and surjective as $h(G) = 1$. It is moreover continuous and open (as $G(\mathbb{R}) \times G(\widehat{\mathbb{Z}})$ is), hence a homeomorphism.

Let us check that $G(\mathbb{R})/G(\mathbb{Z})$ is connected. Observe that

$$G(\mathbb{R}) = s(G_{\text{sc}}(\mathbb{R}))T(\mathbb{R}).$$

But $G_{\text{sc}}(\mathbb{R})$ is connected as G_{sc} is connected and simply connected. We conclude as $T(\mathbb{Z}) \subset G(\mathbb{Z})$ meets every connected component of $T(\mathbb{R})$, since T is \mathbb{Z} -split. The last assertion follows from the following simple fact applied to $A = Z(G)$: if A is a finite multiplicative \mathbb{Z} -group scheme, then the natural map $A(\mathbb{Z}) \rightarrow A(\mathbb{R})$ is bijective (reduce to the case $A = \mu_n$). \square

When $G = \text{PGSp}(2g)$ or $\text{Sp}(2g)$, the cuspidal automorphic representations π of G such that π_∞ is a holomorphic discrete series representation are closely related to vector valued Siegel cuspforms : see e.g. [AS].

A semisimple \mathbb{Z} -group G will be said *definite* if $G(\mathbb{R})$ is compact. This is somewhat the opposite case of Chevalley groups, but a case of great interest in this paper. For instance $\text{SO}(n)$ is definite and there is a natural bijection

$$(6) \quad \text{Cl}(\text{SO}(n)) \xrightarrow{\sim} \tilde{X}_n,$$

because the rank n definite quadratic forms over \mathbb{Z} form a single genus, as recalled in §3.1. If G is definite, then

$$\mathcal{L}(G) = \mathcal{L}_{\text{disc}}(G)$$

by the Peter-Weyl theorem, and the discrete automorphic representations of G have very simple models. Automorphic forms for definite semisimple \mathbb{Z} -groups are a special case of "algebraic modular forms" in the sense of Gross [Gr2].

Proposition 3.6. *Let G be a semisimple definite \mathbb{Z} -group and let (ρ, V) be an irreducible continuous representation of $G(\mathbb{R})$. The vector space $\text{Hom}_{G(\mathbb{R})}(V, \mathcal{L}(G))$ is canonically isomorphic to the space of covariant functions*

$$M_\rho(G) = \{f : G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}}) \rightarrow V^\vee, f(\gamma g) = {}^t\rho(\gamma)^{-1}f(g) \ \forall \gamma \in G(\mathbb{Q}), g \in G(\mathbb{A}_f)\}.$$

In particular, $\dim(M_\rho(G)) = \sum_{\pi \in \Pi_{\text{disc}}(G), \pi_\infty \simeq V} m(\pi).$

The canonical bijection of the statement is $\varphi \mapsto (g \mapsto (v \mapsto \varphi(v)(1 \times g)))$, where $\varphi \in \text{Hom}_{G(\mathbb{R})}(V, \mathcal{L}(G))$, $v \in V$ and $g \in G(\mathbb{A}_f)$. If $g_1, \dots, g_{h(G)} \in G(\mathbb{A}_f)$ are representatives for the classes in $\text{Cl}(G)$, the evaluation map $f \mapsto (f(g_i))$ defines thus a bijection

$$M_\rho(G) \xrightarrow{\sim} \prod_{i=1}^{h(G)} (V^\vee)^{\Gamma_i}$$

where Γ_i is the finite group $G(\mathbb{R}) \cap g_i^{-1}G(\mathbb{Q})g_i$. In particular, to compute $M_\rho(G)$ we are reduced to compute invariants of the finite group $\Gamma_i \subset G(\mathbb{R})$ in the representation V , what we have already studied in Chapter 2. Indeed, the group $G(\mathbb{R})$ is always connected (Chevalley).

Of course if $g_i = 1$, then $\Gamma_i = G(\mathbb{Z})$. In the example of the group $G = \text{SO}(n)$, if $L_i \in \mathcal{L}_n$ is the lattice corresponding to g_i via the bijection (6), then $\Gamma_i = \text{SO}(L_i)$. Later, we will study in details the cases $G = \text{SO}(n)$ where $n = 7, 8$ and 9 , and the definite semisimple \mathbb{Z} -group G_2 .

3.7. Langlands parameterisation of $\Pi_{\text{disc}}(G)$. Let G be any semisimple \mathbb{Z} -group. As G is reductive over \mathbb{Z} , and as each non trivial number field has a ramified prime, the based root datum of $G_{\overline{\mathbb{Q}}}$ has a trivial action of the absolute Galois group of \mathbb{Q} , i.e. the \mathbb{Q} -group $G_{\mathbb{Q}}$ is an inner form of a unique split Chevalley group (see [Gr1]). In particular, the Langlands dual group of G is a complex semisimple algebraic group \widehat{G} (equipped with an isomorphism between the dual based root datum of \widehat{G} and the based root datum of $G_{\overline{\mathbb{Q}}}$). The group \widehat{G} itself is well defined up to inner automorphism. When G is either $\text{PGL}(n)$, $\text{Sp}(2n)$ or of the form SO_L for L a quadratic form of rank n over \mathbb{Z} , \widehat{G} is well-known to be respectively

$$\text{SL}(n, \mathbb{C}), \text{SO}(2n+1, \mathbb{C}), \text{Sp}(n-1, \mathbb{C}) \text{ (} n \text{ odd) or } \text{SO}(n, \mathbb{C}) \text{ (} n \text{ even)}.$$

If $G = \text{SO}_L$, then both $G_{\mathbb{Q}}$ and the $G(\mathbb{A}_f)$ -conjugacy class of $G(\widehat{\mathbb{Z}}) \subset G(\mathbb{A}_f)$ only depend on $L \otimes \mathbb{R}$. We would thus loose nothing in assuming once and for all that L is the standard quadratic form of that signature.

If H is the group of \mathbb{C} -points of any complex semisimple algebraic group over \mathbb{C} , we shall denote by

$$\mathcal{X}(H)$$

the set of collections (c_v) indexed by the places v of \mathbb{Q} , where each c_p (resp. c_∞) is a semisimple conjugacy class in H (resp. $\text{Lie}_{\mathbb{C}}(H)$). By Langlands' interpretation of work of Harish-Chandra and Satake, we have a natural parameterization map

$$c : \Pi(G) \longrightarrow \mathcal{X}(\widehat{G}), \quad \pi \mapsto (c_p(\pi)),$$

where $c_\infty(\pi)$ the infinitesimal character of π_∞ and each $c_p(\pi)$ is Langlands-Satake parameter of π_p . When G is a classical semisimple group over \mathbb{Z} , Arthur's classification describes $\Pi_{\text{disc}}(G)$ in terms of the $\Pi_{\text{cusp}}(\text{PGL}(m))$ for various m 's.

3.8. Arthur's symplectic-orthogonal alternative. By a classical semisimple group over \mathbb{Z} we shall mean either $\mathrm{Sp}(2g)$ for $g \geq 1$ or SO_L for L a standard quadratic form over \mathbb{Z} of rank $\neq 2$ (in rank 1 one obtains the trivial group). The classical Chevalley groups are the groups in this list which are split over \mathbb{Z} , namely $\mathrm{Sp}(2g)$ and the $\mathrm{SO}(p, q)$ with $p - q \in \{0, 1\}$. The definite classical semisimple groups over \mathbb{Z} are the $\mathrm{SO}(n)$, $n \equiv 0, \pm 1 \pmod 8$.

If G is a classical semisimple group over \mathbb{Z} , we shall denote by

$$\mathrm{St} : \widehat{G} \hookrightarrow \mathrm{SL}(n, \mathbb{C})$$

the standard representation of its dual group, which defines in particular the integer $n = n(G)$. This group homomorphism defines in particular a natural map $\mathcal{X}(\widehat{G}) \rightarrow \mathcal{X}(\widehat{\mathrm{PGL}(n)})$ that we shall still denote by St .

Theorem* 3.9. (*Arthur*) *For any $n \geq 1$ and any given self-dual $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n))$ there is a unique classical Chevalley group G^π with $n(G^\pi) = n$ such that there exists $\pi' \in \Pi_{\mathrm{disc}}(G^\pi)$ satisfying $\mathrm{St}(c(\pi')) = c(\pi)$.*

This is [A3, Thm. 1.4.1]. As $n(G^\pi) = n$, the only possibilities for G^π are thus $G^\pi = \mathrm{Sp}(n-1)$ if n is odd, and $G^\pi = \mathrm{SO}(\frac{n}{2}, \frac{n}{2} - 1)$ or $G^\pi = \mathrm{SO}(\frac{n}{2}, \frac{n}{2})$ if n is even. This forces $G^\pi = \mathrm{SO}(2, 1) = \mathrm{PGL}(2)$ if $n = 2$.

As self-dual $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n))$ will be said *orthogonal* (resp. *symplectic*) if $\widehat{G}^\pi = \mathrm{SO}(n, \mathbb{C})$ (resp. $\widehat{G}^\pi = \mathrm{Sp}(n, \mathbb{C})$). For short, we shall define

$$s(\pi) \in \{\pm 1\}$$

to be 1 if π is orthogonal, -1 otherwise. If n is odd then π is necessarily orthogonal, i.e. $s(\pi) = 1$. Arthur's Theorem 3.9 is actually more precise at the infinite place. Indeed, let

$$\mathrm{L}(\pi_\infty) : \mathrm{W}_{\mathbb{R}} \longrightarrow \mathrm{SL}(n, \mathbb{C})$$

be the Langlands parameter of π_∞ . Arthur shows that $\mathrm{L}(\pi_\infty)$ maybe conjugated into $\mathrm{St}(\widehat{G}^\pi) \subset \mathrm{SL}(n, \mathbb{C})$ ([A3, Thm. 1.4.2]). Note that the \widehat{G}^π -conjugacy class of the resulting Langlands parameter

$$\widetilde{\mathrm{L}}(\pi_\infty) : \mathrm{W}_{\mathbb{R}} \longrightarrow \widehat{G}^\pi$$

is not quite canonical, but so is its $\mathrm{Out}(\widehat{G}^\pi)$ -orbit.

Corollary* 3.10. *Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n))$ satisfy conditions (i), (ii) and (iii) of the introduction, and let (w_i) be the Hodge numbers of π .*

- (i) *The w_i are all congruent modulo 2.*
- (ii) *π is orthogonal (resp. symplectic) in the sense above if and only if it is so in the sense of the introduction.*
- (iii) *π is orthogonal if w_1 is even and symplectic otherwise.*

Proof — By definition, $\mathrm{L}(\pi_\infty) : \mathrm{W}_{\mathbb{R}} \rightarrow \mathrm{SL}(n, \mathbb{C})$ is a direct sum of distinct irreducible self-dual representations of $\mathrm{W}_{\mathbb{R}}$. It follows that if $\mathrm{L}(\pi_\infty)$ preserves a non-degenerate pairing on \mathbb{C}^n then each irreducible subspace is non-degenerate as well. Moreover, the

2-dimensional representation I_w has determinant ε^{w+1} and may be conjugate into $O(2, \mathbb{C})$ if and only if w is even. The result follows as $L(\pi_\infty)$ may be conjugate into $\widehat{\text{St}(\widehat{G}^\pi)}$ by the aforementioned result of Arthur. \square

3.11. Arthur's classification : global parameters. Let G be a classical semisimple group over \mathbb{Z} and let $n = n(G)$. Define $s(G) \in \{\pm 1\}$ by $s(G) = 1$ if \widehat{G} is a special orthogonal group, -1 otherwise. Denote by $\Psi_{\text{glob}}(G)$ the set of quadruples

$$(k, (n_i), (d_i), (\pi_i))$$

where $1 \leq k \leq n$ is an integer, where for each $1 \leq i \leq k$ then $n_i \geq 1$ is an integer and d_i is a divisor of n_i , and where $\pi_i \in \Pi_{\text{cusp}}(\text{PGL}(n_i/d_i))$ is self-dual, such that :

- (i) $\sum_{i=1}^k n_i = n$,
- (ii) for each i , $s(\pi_i)(-1)^{d_i+1} = s(G)$,
- (iii) if $i \neq j$ and $(n_i, d_i) = (n_j, d_j)$ then $\pi_i \neq \pi_j$.

The set $\Psi_{\text{glob}}(G)$ only depends on $n(G)$ and $s(G)$. Two elements $(k, (n_i), (d_i), (\pi_i))$ and $(k', (n'_i), (d'_i), (\pi'_i))$ in $\Psi_{\text{glob}}(G)$ are said equivalent if $k = k'$ and if there exists $\sigma \in \mathfrak{S}_k$ such that $n'_i = n_{\sigma(i)}$, $d'_i = d_{\sigma(i)}$ and $\pi'_i = \pi_{\sigma(i)}$ for each i . An element of $\Psi_{\text{glob}}(G)$ will be called a global Arthur parameter for G . The class $\underline{\psi}$ of $\psi = (k, (n_i), (d_i), (\pi_i))$ will also be denoted symbolically by

$$\underline{\psi} = \pi_1[d_1] \oplus \pi_2[d_2] \oplus \cdots \oplus \pi_k[d_k].$$

In the witting above we shall replace the symbol $\pi_i[d_i]$ by $[d_i]$ if $n_i = d_i$ (as then π_i is the trivial representation), and by π_i if $d_i = 1$ and $n_i \neq d_i$.

Let $\psi \in \Psi_{\text{glob}}(G)$. Recall that for each integer $d \geq 1$, the \mathbb{C} -group $\text{SL}(2)$ has a unique irreducible \mathbb{C} -representation ν_d of dimension d , namely $\text{Sym}^{d-1}(\mathbb{C}^2)$. Condition (i) allows to define a morphism

$$\rho_\psi : \prod_{i=1}^k \text{SL}(n_i/d_i) \times \text{SL}(2) \longrightarrow \text{SL}(n)$$

(canonical up to conjugation by $\text{SL}(n, \mathbb{C})$) obtained as the direct sum of the representations $\mathbb{C}^{n_i/d_i} \otimes \nu_{d_i}$. One obtains this way a canonical map

$$\rho_\psi : \prod_{i=1}^k \mathcal{X}(\text{SL}(n_i/d_i)) \times \mathcal{X}(\text{SL}(2)) \longrightarrow \mathcal{X}(\text{SL}(n)).$$

A specific element of $\mathcal{X}(\text{SL}(2))$ plays an important role in Arthur's theory : it is the element $e = (e_v)$ defined by $e_p = \text{diag}(p^{-1/2}, p^{1/2})$ (positive square roots) for each prime p , and by $e_\infty = \text{diag}(1/2, -1/2)$. As is well known, $e = c(1)$ where $1 \in \Pi_{\text{disc}}(\text{PGL}(2))$ is the trivial representation.

Theorem* 3.12. (*Arthur's classification*) Let G be any classical semisimple group over \mathbb{Z} and let $\pi \in \Pi_{\text{disc}}(G)$. There is a $\psi(\pi) = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{glob}}(G)$ unique up to equivalence such that

$$\text{St}(c(\pi)) = \rho_\psi \left(\prod_{i=1}^k c(\pi_i) \times e \right).$$

When G is a Chevalley group this follows from [A3, Thm. 1.5.2], otherwise it is expected to be part of the forthcoming last chapter in *loc.cit.* The part of the theorem concerning the infinitesimal character is a property of Shelstad's transfert (see [Sh2, Lemma 15.1], [Me, Lemma 25].)

Definition 3.13. The global Arthur parameter $\psi(\pi)$ will be called the global Arthur parameter of π .

For instance if $1_G \in \Pi_{\text{disc}}(G)$ denotes the trivial representation of G , then it is well-known that the Arthur parameter of 1_G is $[n(G)]$, unless $\widehat{G} = \text{SO}(2m, \mathbb{C})$ in which case it is $[1] \oplus [n(G) - 1]$.

Let $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{glob}}(G)$. The associated triple $(k, (n_i, d_i))$, taken up to permutations of the (n_i, d_i) , will be called the *endoscopic type* of ψ . Moreover, one usually says that ψ is *stable* if $k = 1$ and *endoscopic* otherwise. The generalized Ramanujan conjecture asserts that each π_i is tempered. We shall thus say that ψ is *tempered* if $d_i = 1$ for all i . If $\psi = \psi(\pi)$, the Ramanujan conjecture asserts then that π is tempered if and only if $\psi(\pi)$ is. In some important cases, e.g. the special case where $G(\mathbb{R})$ is compact, this conjecture is actually known in most cases (see below). We will say that π is *stable*, *endoscopic* or *formally tempered* if $\psi(\pi)$ is respectively *stable*, *endoscopic* or *tempered*. We will also talk about the endoscopic type of a π for the endoscopic type of $\psi(\pi)$.

Our last task is to explain Arthur's converse to the theorem above, namely to decide whether a given $\psi \in \Psi_{\text{glob}}(G)$ is in the image of the map $\pi \mapsto \psi(\pi)$. This is the content of the so-called Arthur's multiplicity formula. Our aim until the end of this chapter will be to state certain special cases of this formula.

3.14. The packet $\Pi(\psi)$ of a $\psi \in \Psi_{\text{glob}}(G)$. Fix $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{glob}}(G)$. If p is a prime number, define

$$\Pi_p(\psi)$$

as the set of isomorphism classes of $G(\mathbb{Z}_p)$ -spherical irreducible smooth representations of $G(\mathbb{Q}_p)$ whose Satake parameter s_p , a semisimple conjugacy class in \widehat{G} , satisfies

$$\text{St}(s_p) = \rho_\psi \left(\prod_{i=1}^k c_p(\pi_i) \times e_p \right).$$

This relation uniquely determines the $\text{Out}(\widehat{G})$ -orbit of s_p . It follows that $\Pi_p(\psi)$ is a singleton, unless $\widehat{G} \simeq \text{SO}(2m, \mathbb{C})$ and $\text{St}(s_p)$ does not possess the eigenvalue ± 1 (so each n_i is even), in which case it has 2 elements.

We shall now associate to ψ a $\text{Out}(\widehat{G})$ -orbit of equivalence classes of archimedean Arthur parameters for $G_{\mathbb{R}}$, which will eventually lead in some cases to a definition of a set $\Pi_{\infty}(\psi)$ of irreducible unitary representations of $G(\mathbb{R})$. Denote by $\Psi(G_{\mathbb{R}})$ the set of such parameters, i.e. of continuous homomorphisms

$$\psi_{\mathbb{R}} : W_{\mathbb{R}} \times \text{SL}(2, \mathbb{C}) \longrightarrow \widehat{G}$$

which are \mathbb{C} -algebraic on the $\text{SL}(2, \mathbb{C})$ -factor and such that the image of any element of $W_{\mathbb{R}}$ is semisimple. Two such parameters are said equivalent if they are conjugate under \widehat{G} . An important invariant of an equivalence class of parameters $\psi_{\mathbb{R}}$ is its *infinitesimal character*

$$z_{\psi_{\mathbb{R}}}$$

which is a semisimple conjugacy class in $\mathfrak{g}_{\mathbb{C}}$ given according to a recipe of Arthur : see e.g. §10.2 for the general definition. It is also the infinitesimal character of the Langlands parameter associated by Arthur to $\psi_{\mathbb{R}}$.

We now go back to the global Arthur parameter ψ . By assumption (ii) on ψ , each space $\mathbb{C}^{n_i/d_i} \otimes \nu_{d_i}$ carries a natural representation of $\widehat{G}^{\pi_i} \times \text{SL}(2, \mathbb{C})$ which preserves a non-degenerate bilinear form unique up to scalars, which is symmetric if $s(\widehat{G}) = 1$ and antisymmetric otherwise. One thus obtains a \mathbb{C} -morphism

$$r_{\psi} : \prod_{i=1}^k \widehat{G}^{\pi_i} \times \text{SL}(2, \mathbb{C}) \longrightarrow \widehat{G}.$$

The collection of $\widetilde{L}(\pi_i) : W_{\mathbb{R}} \rightarrow \widehat{G}^{\pi_i}$ defines by composition with r_{ψ} a morphism

$$\psi_{\infty} : W_{\mathbb{R}} \times \text{SL}(2, \mathbb{C}) \rightarrow \widehat{G},$$

which is by definition the archimedean Arthur parameter associated to ψ . The $\text{Out}(\widehat{G})$ -orbit of the equivalence class of ψ_{∞} only depends on ψ . In particular, only the $\text{Out}(\widehat{G})$ -orbit of its infinitesimal character is well-defined, with this caveat in mind we shall still denote it by $z_{\psi_{\infty}}$. By definition we have

$$(7) \quad \text{St}(z_{\psi_{\infty}}) = \rho_{\psi} \left(\prod_{i=1}^k c_{\infty}(\pi_i) \times e_{\infty} \right),$$

which determines $z_{\psi_{\infty}}$ uniquely.

Consider the following two properties of an Arthur parameter $\psi_{\mathbb{R}} \in \Psi(G_{\mathbb{R}})$:

- (a) $z_{\psi_{\mathbb{R}}}$ is the infinitesimal character of a finite dimensional \mathbb{C} -representation of $G(\mathbb{C})$,
- (b) $\text{St} \circ \psi_{\mathbb{R}}$ is a multiplicity free representation of $W_{\mathbb{R}} \times \text{SL}(2, \mathbb{C})$.

Moreover, if ψ satisfies (a) (resp (b)) then so does $\tau \circ \psi$ where $\tau \in \text{Aut}(\widehat{G})$. In particular, it makes sense to say that ψ_{∞} satisfies (a) if $\psi \in \Psi_{\text{glob}}(G)$.

Definition 3.15. Denote by $\Psi_{\text{alg}}(G) \subset \Psi_{\text{glob}}(G)$ the subset of ψ such that ψ_{∞} satisfies (a).

By the purity lemma [Cl, Lemme 4.9], observe moreover that if $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{alg}}(G)$, then $(\pi_i)_\infty$ is tempered for each i . The following lemma follows easily :

Lemma 3.16. *Let $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{alg}}(G)$.*

- (i) *Then ψ_∞ satisfies (b) as well, except perhaps in the following very specific case : \widehat{G} is an even orthogonal group, $n(G) \equiv 2 \pmod{4}$, and $\text{St}(z_{\psi_\infty})$ possesses the eigenvalue 0.*
- (ii) *Moreover, if ψ_∞ satisfies (a) and (b) then each π_i satisfies condition (iii) of the introduction (as well as (i) and (ii) by definition).*

This is the first important motivation for the consideration of the properties (a) and (b). The second is that if $G(\mathbb{R})$ is compact, and if $\pi \in \Pi_{\text{disc}}(G)$, then $\psi(\pi)_\infty$ obviously satisfies (a), as well as (b) because $n(G) \equiv 0, -1, +1 \pmod{8}$.

Corollary* 3.17. *Assume that $\widehat{G} \neq \text{SO}(4m+2, \mathbb{C})$. If $\pi \in \Pi_{\text{disc}}(G)$ is such that π_∞ has the infinitesimal character of a finite dimensional \mathbb{C} -representation of $G(\mathbb{C})$, and if $\psi(\pi) = (k, (n_i), (d_i), (\pi_i))$, then each π_i satisfies conditions (i), (ii) and (iii) of the introduction.*

In particular, each π_i satisfies the Ramanujan conjecture, unless perhaps $\widehat{G} = \text{SO}(2m, \mathbb{C})$ and $\text{St}(z_{\psi_\infty})$ contains the eigenvalue 0.

We shall exclude from now on the particular case $s(G) = 1$ and $n(G) \equiv 2 \pmod{4}$, i.e. we assume that

$$\widehat{G} \neq \text{SO}(4m+2, \mathbb{C}).$$

We already said that $G(\mathbb{R})$ is an inner form of a split group. As $\widehat{G} \neq \text{SO}(4m+2, \mathbb{C})$, it is also an inner form of a compact group (this is of course obvious if $G(\mathbb{R})$ is already compact). A parameter $\psi_{\mathbb{R}} \in \Psi(G_{\mathbb{R}})$ satisfying conditions (a) and (b) above is called an Adams-Johnson parameter for $G_{\mathbb{R}}$. The set of these parameters is denoted by

$$\Psi_{\text{AJ}}(G_{\mathbb{R}}) \subset \Psi(G_{\mathbb{R}}).$$

We refer to the Appendix for a general discussion about them. For $\psi_{\mathbb{R}} \in \Psi_{\text{AJ}}(G_{\mathbb{R}})$, Adams and Johnson have defined in [AdJ] a finite set $\Pi(\psi_{\mathbb{R}})$ of (cohomological) irreducible unitary representations of $G(\mathbb{R})$. In the notations of the appendix, the group $G(\mathbb{R})$ is isomorphic to a group of the form G_t for some $t \in \mathcal{X}_1(T)$. Recall that up to inner isomorphisms, G_t only depends on the W -orbit of $tZ(G)$. We fix such an isomorphism between $G(\mathbb{R})$ and $G_{[t]}$ and set $\Pi(\psi_{\mathbb{R}}) = \Pi(\psi, G_{[t]})$. As $\text{Aut}(G(\mathbb{R})) \neq \text{Int}(G(\mathbb{R}))$ in general, this choice of an isomorphism might be problematic in principle. However, a simple case-by-case inspection shows that for any classical semisimple \mathbb{Z} -group G the natural map $\text{Out}(G) \rightarrow \text{Out}(G(\mathbb{R}))$ is surjective, so that this choice virtually plays no role in the following considerations. We shall say more about this when we come to the multiplicity formula.

Let $\psi \in \Psi_{\text{alg}}(G)$. If $\text{Out}(\widehat{G}) = 1$, or more generally if the $\text{Out}(\widehat{G})$ -orbit of the equivalence class of ψ_∞ has one element, we set

$$\Pi_\infty(\psi) = \Pi(\psi_\infty).$$

In the remaining case, we define $\Pi_\infty(\psi)$ as the disjoint union of the two sets $\Pi(\psi_\mathbb{R})$ where $\psi_\mathbb{R}$ is an equivalence class of parameters in the $\text{Out}(\widehat{G})$ -orbit of ψ_∞ . Recall from §10.7 that the isomorphism $G(\mathbb{R}) \rightarrow G_t$ fixed above furnishes a canonical parameterization map

$$\tau : \Pi_\infty(\psi) \longrightarrow \text{Hom}(C_{\psi_\infty}, \mathbb{C}^*).$$

The presence of C_{ψ_∞} in the target, rather than S_{ψ_∞} , follows from the fact that $G(\mathbb{R})$ is a pure inner form of a split group and from Lemma 10.14. When the $\text{Out}(\widehat{G})$ -orbit of the equivalence class of ψ_∞ has two elements, say ψ_1, ψ_2 , there is a canonical way of identifying C_{ψ_1} and C_{ψ_2} , thus it is harmless to denote them by the same name C_{ψ_∞} .

Definition 3.18. *If $\psi \in \Psi_{\text{alg}}(G)$ set $\Pi(\psi) = \{\pi \in \Pi(G), \pi_v \in \Pi_v(\psi) \forall v\}$.*

The first conjecture we are in position to formulate is a comparison between the Arthur packet attached to a $\psi_\mathbb{R} \in \Psi_{\text{AJ}}(G_\mathbb{R})$, as defined in his book [A3, §2.2] by twisted endoscopy when $G_\mathbb{R}$ is split, and the packet $\Pi(\psi_\mathbb{R})$ of Adams and Johnson recalled above (in a slightly weak sense in the case $\widehat{G} = \text{SO}(2r, \mathbb{C})$). It seems widely believed that they indeed coincide, although no proof seems to have been given yet. This is the first assumption in our work. A first consequence would be the following conjecture. Observe that this conjecture is obvious when $G(\mathbb{R})$ is compact.

Conjecture 3.19. *If $\pi \in \Pi_{\text{disc}}(G)$ and if π_∞ has the infinitesimal character of a finite dimensional \mathbb{C} -representation of $G(\mathbb{C})$ then $\pi \in \Pi(\psi(\pi))$.*

So far we have defined for each $\psi \in \Psi_{\text{alg}}(G)$ a set $\Pi(\psi)$ as well as a parameterization τ of $\Pi_\infty(\psi)$. This set is e.g. a singleton when $G = \text{SO}(n)$ with n odd, and it is finite, in bijection with $\Pi_\infty(\psi)$, if $\text{Out}(\widehat{G}) = 1$. Arthur's multiplicity formula is a formula for $m(\pi)$ for each $\pi \in \Pi(\psi)$, at least when $\text{Out}(\widehat{G}) = 1$. This formula contains a last ingredient that we now study.

3.20. The character ε_ψ of C_ψ . Consider some $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{alg}}(G)$ and denote by C_ψ the centraliser of $\text{Im}(r_\psi)$ in \widehat{G} . This is an elementary abelian 2-group that we may describe as follows. Observe that by property (a) on z_{ψ_∞} each n_i is even, except perhaps exactly one or two of them when \widehat{G} is an orthogonal group. If $1 \leq i \leq k$ is such that n_i is even, then $Z(\widehat{G^{\pi_i}}) \simeq \mathbb{Z}/2\mathbb{Z}$, and we denote by

$$s_i \in C_\psi$$

the generator of $r_\psi(Z(\widehat{G^{\pi_i}}))$. The next lemma is clear.

Lemma 3.21. *C_ψ is generated by $Z(\widehat{G})$ and by the s_i .*

A first important ingredient in Arthur's multiplicity formula is Arthur's character

$$\varepsilon_\psi : C_\psi \longrightarrow \{\pm 1\}.$$

It has been defined by Arthur in full generality in [A1]. We shall apply formula (1.5.6) of [A3]. By definition, ε_ψ is trivial on $Z(\widehat{G}) \subset C_\psi$. In the special case here, we thus only

have to give the $\varepsilon_\psi(s_i)$. As the representation $\nu_a \otimes \nu_b$ of $\mathrm{SL}(2, \mathbb{C})$ has exactly $\mathrm{Min}(a, b)$ irreducible factors, the formula loc. cit. is easily seen to be

$$(8) \quad \varepsilon_\psi(s_i) = \prod_{j \neq i} \varepsilon(\pi_i \times \pi_j)^{\mathrm{Min}(d_i, d_j)}$$

where $\varepsilon(\pi_i \times \pi_j) = \pm 1$ is the sign such that $L(1-s, \pi_i \times \pi_j) = \varepsilon(\pi_i \times \pi_j) L(s, \pi_i \times \pi_j)$ (completed L -functions). An important result of Arthur asserts that $\varepsilon(\pi_i \times \pi_j) = 1$ if $s(\pi_i)s(\pi_j) = 1$, so that in the product above we may restrict to the j such that $s(\pi_j) \neq s(\pi_i)$.

As the cuspidal automorphic representations π_i are unramified at each finite place, and also quite specific at the infinite place by Lemma 3.16 (ii), one obtains an explicit formula for this number in terms of the Hodge numbers of the π_i . The precise formula is as follows. There is a unique collection of complex numbers

$$\varepsilon(r) \in \{1, i, -1, -i\}$$

defined for all the isomorphism classes of continuous representations $r : W_{\mathbb{R}} \rightarrow \mathrm{GL}(m, \mathbb{C})$ which are trivial on $\mathbb{R}_{>0} \subset W_{\mathbb{C}}$, such that :

- (i) $\varepsilon(r \oplus r') = \varepsilon(r)\varepsilon(r')$ for all r, r' ,
- (ii) $\varepsilon(\mathrm{I}_w) = i^{w+1}$ for any integer $w \geq 0$,
- (iii) $\varepsilon(1) = 1$.

For instance, if $w, w' \geq 0$ are integers, one has

$$\varepsilon(\mathrm{I}_w \otimes \mathrm{I}_{w'}) = (-1)^{1+\mathrm{Max}(w, w')},$$

as $\mathrm{I}_w \otimes \mathrm{I}_{w'} \simeq \mathrm{I}_{w+w'} \oplus \mathrm{I}_{|w-w'|}$. Moreover $\mathrm{I}_w \otimes \varepsilon_{\mathbb{C}/\mathbb{R}} = \mathrm{I}_w$ if $\varepsilon_{\mathbb{C}/\mathbb{R}}$ denotes the sign of $W_{\mathbb{R}}$.

If $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n))$ and $\pi' \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n'))$ both satisfy the assumptions (i), (ii) and (iii) of the introduction, then both $L(\pi_\infty)$ and $L(\pi'_\infty)$ are trivial on $\mathbb{R}_{>0}$, and one has

$$(9) \quad \varepsilon(\pi \times \pi') = \varepsilon(L(\pi_\infty) \otimes L(\pi'_\infty)).$$

See [Ta, §4], [A3, §1.3], and Cogdell's lectures [Cog, Ch. 4] for a survey. This allows to compute ε_ψ in all cases. See [ChLa] for some explicit formulas.

3.22. Arthur's multiplicity formula. Let G be a classical semisimple group over \mathbb{Z} such that $\widehat{G} \neq \mathrm{SO}(4m+2, \mathbb{C})$ and let $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{alg}}(G)$. Following Arthur, set

$$m_\psi = \begin{cases} 2 & \text{if } s(\widehat{G}) = 1 \text{ and } n_i \equiv 0 \pmod{2} \text{ for all } 1 \leq i \leq k, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the following equivalence relation \sim on $\Pi(G)$. The relation \sim is trivial (i.e. equality) unless \widehat{G} is an even orthogonal group, in which case one may assume that $G = \mathrm{SO}_L$ is a standard even orthogonal group. Consider the outer automorphism s of $G(\mathbb{Q})$ induced by the conjugation by any $s_\alpha \in \mathrm{O}(L)$ as in Remark 3.2. The element s

does not depend on the choice of the root $\alpha \in L$. If $\pi, \pi' \in \Pi(G)$ we define $\pi \sim \pi'$ if $\pi_v \in \{\pi'_v, \pi'_v \circ s\}$ for each v .

For $\pi \in \Pi(G)$, recall that $m(\pi)$ denotes the multiplicity of π in $L_{\text{disc}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. Recall that we have defined a group C_ψ in §3.20, as well as a group C_{ψ_∞} in §3.14. By definition there is a canonical inclusion

$$C_\psi \subset C_{\psi_\infty}.$$

Conjecture 3.23. (*Arthur's multiplicity formula*) Let $\psi = \{(k, (n_i), (d_i), (\pi_i))\} \in \Psi_{\text{alg}}(G)$ and let $\pi \in \Pi(\psi)$. Then

$$\sum_{\pi' \in \Pi(\psi), \pi' \sim \pi} m(\pi') = \begin{cases} 0 & \text{if } \tau(\pi_\infty) \neq \varepsilon_\psi, \\ m_\psi & \text{otherwise.} \end{cases}$$

Observe that $\{\pi' \in \Pi(\psi), \pi \sim \pi'\}$ is the singleton $\{\pi\}$ unless \widehat{G} is an even orthogonal group.

At the moment this multiplicity formula is still conjectural. When G is a Chevalley group, it is a theorem by the results of [A3] if we replace the parameterized set $(\Pi_\infty(\psi), \tau)$ above by the ones defined by Arthur in [A3]. More accurately, it depends on the same assumptions as in the work [A3]. The multiplicity formula even holds then for all the global parameters $\psi \in \Psi_{\text{glob}}(G)$. The case of a general G has also been announced by Arthur : see Chap. 9 of loc. cit.

In this paper, we shall use this conjecture only in the following list of special cases. In each case we will explicit completely the multiplicity formula in terms of the Hodge numbers of the π_i appearing in ψ . We have already done so for the term ε_ψ in the previous paragraph. In each case we also discuss the dependence of the multiplicity formula on the choice of the identification of $G(\mathbb{R})$ that we have fixed in §3.14 to define $\tau(\pi_\infty)$.

3.23.1. The definite odd orthogonal group $G = \text{SO}(2r + 1)$.

In this case $r \equiv 0, 3 \pmod{4}$ and $\widehat{G} = \text{Sp}(2r, \mathbb{C})$. Consider the standard based root datum for $(\widehat{G}, \widehat{B}, \widehat{T})$ with $X^*(\widehat{T}) = \mathbb{Z}^r$ with canonical basis (e_i) and

$$\Phi^+(\widehat{G}, \widehat{T}) = \{\pm 2e_i, \pm e_i \pm e_j, 1 \leq i < j \leq r\}.$$

We conjugate r_ψ in \widehat{G} so that the centralizer of $\varphi_{\psi_\infty}(W_{\mathbb{C}})$ is \widehat{T} , and that $\varphi_{\psi_\infty}(z) = z^\lambda \bar{z}^{\lambda'}$ with $\lambda \in \frac{1}{2}X_*(\widehat{T})$ dominant with respect to \widehat{B} .

There is a unique element in $\Pi_\infty(\psi)$, namely the irreducible representation with infinitesimal character z_{ψ_∞} . The character $\tau(\pi_\infty)$ is absolutely canonical here as each automorphism of $G(\mathbb{R})$ is inner and there is a unique choice of strong real form t for G_t (namely $t = 1$). By Cor. 10.12, this character $\tau(\pi_\infty)$ is $(\rho^\vee)_{|C_{\psi_\infty}}$, where ρ^\vee denotes the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$, namely $\rho^\vee = re_1 + (r-1)e_2 + \cdots + e_r$. In particular $\rho^\vee \in X^*(\widehat{T})$ and it satisfies the congruence

$$\rho^\vee \equiv e_r + e_{r-2} + e_{r-4} + \cdots \pmod{2X^*(\widehat{T})}.$$

Observe that $\rho^\vee(-1) = 1$ as $r \equiv 0, 3 \pmod{4}$, so that ρ^\vee is trivial on $Z(\widehat{G})$.

Consider the generators s_i of C_ψ introduced in §3.20. We shall now give an explicit formula for the $\rho^\vee(s_i)$. Fix some $i \in \{1, \dots, k\}$ and write $n_i = r_i d_i$. Observe first that if d_i is even, it is clear that

$$\rho^\vee(s_i) = (-1)^{\frac{d_i}{2}[(r_i+1)/2]}.$$

If d_i is odd, in which case r_i is even, the sign $\rho^\vee(s_i)$ depends on the Hodge numbers of π_i . Precisely, denote by

$$w_1 > \dots > w_r$$

the positive odd integers w_j such that the eigenvalues of $\text{St}(z_{\psi_\infty})$ in $\text{SL}(2r, \mathbb{C})$ are the $\pm w_j$ (see formula (7)). There is a unique subset $J \subset \{1, \dots, r\}$ such that the Hodge numbers of π_i are the w_j for $j \in J$. Denote by J' the subset of $j \in J$ such that $j \equiv r \pmod{2}$. It is then clear that

$$\rho^\vee(s_i) = (-1)^{|J'|}.$$

3.23.2. The definite even orthogonal group $G = \text{SO}(2r)$.

In this case $r \equiv 0 \pmod{4}$ and $\widehat{G} = \text{SO}(2r, \mathbb{C})$. Consider the standard based root datum for $(\widehat{G}, \widehat{B}, \widehat{T})$ with $X^*(\widehat{T}) = \mathbb{Z}^r$ with canonical basis (e_i) and

$$\Phi^+(\widehat{G}, \widehat{T}) = \{e_i \pm e_j, 1 \leq i < j \leq r\}.$$

We conjugate r_ψ in \widehat{G} as in the odd orthogonal case.

If the $\text{Out}(\widehat{G})$ -orbit of ψ_∞ consists of only one equivalence class, in which case the $\text{Out}(\widehat{G})$ -orbit of z_{ψ_∞} is a singleton, then the unique element of $\Pi_\infty(\psi)$ is the representation of $G(\mathbb{R})$ with infinitesimal character z_{ψ_∞} . Otherwise, the two elements of $\Pi_\infty(\psi)$, again two finite dimensional irreducible representations, have the property that their infinitesimal characters are exchanged by the outer automorphism of $G(\mathbb{R})$, and both in the $\text{Out}(\widehat{G})$ -orbit of z_{ψ_∞} . Observe that there is still the possibility that the $\text{Out}(\widehat{G})$ -orbit of z_{ψ_∞} is a singleton : in this case $\Pi_\infty(\pi)$ consists of two isomorphic representations. However, observe also that by definition all the members of $\Pi(\psi) \subset \Pi(G)$ have the same archimedean component in this case.

Recall we have fixed an isomorphism between $G(\mathbb{R})$ and G_t for $t = \{\pm 1\} \in Z(G)$ as in §10.1. Assume first that we actually chose $t = 1$. It follows that the one or two elements in $\Pi_\infty(\psi)$ have the same character ρ^\vee by Cor. 10.12. Here we have $\rho^\vee = (r-1)e_1 + (r-2)e_2 + \dots + e_{r-1}$, thus $\rho^\vee \in X^*(\widehat{T})$ and

$$\rho^\vee \equiv e_{r-1} + e_{r-3} + e_{r-5} + \dots \pmod{2X^*(\widehat{T})}.$$

Observe again that $\rho^\vee(-1) = 1$ as $r \equiv 0 \pmod{4}$.

Consider the generators s_i of C_ψ introduced in §3.20. Fix some $i \in \{1, \dots, k\}$ and write $n_i = r_i d_i$. If d_i is even, then r_i is even as well and we have

$$\rho^\vee(s_i) = (-1)^{\frac{n_i}{4}}.$$

If d_i is odd, in which case r_i is even as n_i is even by assumption, the sign $\rho^\vee(s_i)$ depends on the Hodge numbers of π_i . Precisely, denote by

$$w_1 > \cdots > w_r$$

the positive even integers w_j such that the eigenvalues of $\text{St}(z_{\psi_\infty})$ in $\text{SL}(2r, \mathbb{C})$ are the $\pm w_j$ (see formula (7)). There is a unique subset $J \subset \{1, \dots, r\}$ such that the Hodge numbers of π_i are the w_j for $j \in J$. Denote by J' the subset of $j \in J$ such that $j \equiv r - 1 \pmod{2}$. It is then clear that

$$\rho^\vee(s_i) = (-1)^{|J'|}.$$

For coherence reasons, we shall check now that the multiplicity formula does not change if we choose to identify $G(\mathbb{R})$ with G_{-1} or if we modify the fixed isomorphism by the outer automorphism of $G(\mathbb{R})$. This second fact is actually trivial by what we already said, so assume that we identified $G(\mathbb{R})$ with G_{-1} . The effect of this choice is that the one or two elements of $\Pi_\infty(\psi)$ become parameterized by the character

$$\rho^\vee + \chi,$$

where χ is the generator of the group $\mathcal{N}(T)$, by Lemma 10.10. As $-1 = e^{i\pi\chi}$ we have

$$\chi \equiv \sum_{i=1}^r e_i \pmod{2X_*(\widehat{T})}$$

and we claim that this character is trivial on C_ψ . Indeed, it follows from Lemma 3.16 that if n_i is even then $n_i \equiv 0 \pmod{4}$, so that $\chi(s_i) = (-1)^{n_i/2} = 1$.

3.23.3. The Chevalley groups $\text{Sp}(2g)$, $\text{SO}(2, 2)$ and $\text{SO}(3, 2)$.

The case of the symplectic groups $\text{Sp}(2g)$ will be treated in details in Chapter 9, especially in §9.2. We shall only consider there the multiplicity formula for a π such that π_∞ is a holomorphic discrete series.

The cases $G = \text{SO}(2, 2)$ and $\text{SO}(3, 2)$ will be used in Chapter 4. For $\text{SO}(2, 2)$ we shall not use that Arthur's packets are the same as the ones of Adams-Johnson. For $G = \text{SO}(3, 2)$ we shall need it only in §4.2, i.e. to compute $S(w, v)$, for the $\psi \in \Psi_{\text{alg}}(G)$ of the form $\pi \oplus [2]$. In this case this is probably not too difficult to check but due to the already substantial length of this paper we decided not to include this twisted character computation here. We hope to do so in the future.

4. DETERMINATION OF $\Pi_{\text{alg}}^\perp(\text{PGL}(n))$ FOR $n \leq 5$

In this chapter we justify the formulae for $S(w)$ and $S(w, v)$ given in the introduction and prove Theorem 1.8 there. For $n \geq 1$ an integer, denote by

$$\Pi_{\text{cusp}}^\perp(\text{PGL}(n)) \subset \Pi_{\text{cusp}}(\text{PGL}(n))$$

the subset of self-dual π , i.e. such that $\pi^\vee = \pi$. Write

$$\Pi_{\text{cusp}}^\perp(\text{PGL}(n)) = \Pi_{\text{cusp}}^s(\text{PGL}(n)) \coprod \Pi_{\text{cusp}}^o(\text{PGL}(n))$$

where $\pi \in \Pi_{\text{cusp}}^s(\text{PGL}(n))$ if and only if π is symplectic in the sense of Arthur (§ 3.8). For $*$ = o , s or \perp , let us define also

$$\Pi_{\text{alg}}^*(\text{PGL}(n)) \subset \Pi_{\text{cusp}}^*(\text{PGL}(n))$$

as the subset of π satisfying condition (iii) of the introduction. In particular, a $\pi \in \Pi_{\text{alg}}^\perp(\text{PGL}(n))$ has $r = [n/2]$ well defined Hodge numbers $w_1 > \cdots w_r \geq 0$ which are integers all congruent to $\frac{s(\pi)-1}{2} \pmod{2}$.

4.1. Determination of $\Pi_{\text{cusp}}^\perp(\text{PGL}(2))$. A representation $\pi \in \Pi_{\text{cusp}}(\text{PGL}(2))$ is necessarily self-dual, and even symplectic by Theorem 3.9, so that

$$\Pi_{\text{cusp}}(\text{PGL}(2)) = \Pi_{\text{cusp}}^\perp(\text{PGL}(2)) = \Pi_{\text{cusp}}^s(\text{PGL}(2)).$$

If $\pi \in \Pi_{\text{cusp}}(\text{PGL}(2))$ it is well known that π_∞ satisfies condition (iii) for the Hodge number w if and only if it is the unique discrete series representation with infinitesimal character $\text{diag}(w, -w) \in \mathfrak{sl}(2, \mathbb{C})$.

Let $w \geq 1$ be an odd integer and let \mathcal{F}_w be the set of

$$F = \sum_{m \geq 1} a_m q^m \in S_{w+1}(\text{SL}(2, \mathbb{Z}))$$

which are eigenforms for all the Hecke operators and normalized so that $a_1 = 1$ (see [Se1]). As is well-known, \mathcal{F}_w is a basis of the complex vector space $S_{w+1}(\text{SL}(2, \mathbb{Z}))$. Moreover, each $F \in \mathcal{F}_w$ generates a $\pi_F \in \Pi_{\text{cusp}}(\text{PGL}(2))$ which satisfies furthermore (iii) for the integer $w_1 = w$. The map $F \mapsto \pi_F$ is a bijection between \mathcal{F}_w and the set of π in $\Pi_{\text{alg}}(\text{PGL}(2))$ such that π_∞ has Hodge number w . In particular

$$S(w) = \dim(S_{w+1}(\text{SL}(2, \mathbb{Z})))$$

as recalled in the introduction. Moreover, we shall always identify an $F \in \mathcal{F}_w$ with π_F in the bijection above, and simply write $F \in \Pi_{\text{alg}}(\text{PGL}(2)) \subset \Pi_{\text{cusp}}(\text{PGL}(2))$. In particular, for $w \in \{11, 13, 15, 17, 19, 21\}$ we shall denote by

$$\Delta_w \in \Pi_{\text{alg}}(\text{PGL}(2))$$

the unique element with Hodge number w .

4.2. Determination of $\Pi_{\text{alg}}^s(\text{PGL}(4))$.

Fix $w > v$ odd positive integers. Let $S_{w,v}(\text{Sp}(4, \mathbb{Z}))$ be the space of Siegel cusp forms of genus 2 as in the introduction. Denote also by

$$\Pi_{w,v}(\text{PGSp}(4)) \subset \Pi_{\text{cusp}}(\text{PGSp}(4))$$

the subset of $\pi \in \Pi_{\text{cusp}}(\text{PGSp}(4))$ such that π_∞ is the holomorphic discrete series whose infinitesimal character has the eigenvalues $\pm w, \pm v$, viewed as a semisimple conjugacy class in $\mathfrak{sl}(4, \mathbb{C})$. It is well-known that to each Hecke-eigenform F in $S_{w,v}(\text{Sp}(4, \mathbb{Z}))$ one may associate a unique $\pi_F \in \Pi_{w,v}(\text{PGSp}(4))$, and that the image of the map $F \mapsto \pi_F$ is $\Pi_{w,v}(\text{PGSp}(4))$ (see e.g. [AS]).

The semisimple \mathbb{Z} -group $\text{PGSp}(4)$ is isomorphic to $\text{SO}(3, 2)$ hence we may view it as a classical semisimple group over \mathbb{Z} . It follows from Arthur's multiplicity formula that the multiplicity of any such π_F as above is 1, so that the Hecke-eigenspace containing a given Hecke-eigenform F is actually not bigger than $\mathbb{C}F$. It follows that if we denote by $\mathcal{F}_{w,v}$ the set of these (one dimensional) Hecke-eigenspaces in $S_{w,v}(\text{Sp}(4, \mathbb{Z}))$, we thus get

$$|\mathcal{F}_{w,v}| = \dim S_{w,v}(\text{Sp}(4, \mathbb{Z})) = |\Pi_{w,v}(\text{PGSp}(4))|.$$

The following formula was claimed in the introduction.

Proposition 4.3.** *For $w > v > 0$ odd, $S(w, v) = S_{w,v}(\text{Sp}(4, \mathbb{Z})) - \delta_{v=1} \delta_{w \equiv 1 \pmod{4}} S(w)$.*

Before starting the proof, recall that if φ is a discrete series Langlands parameter for $\text{PGSp}(4, \mathbb{R})$, its L -packet $\Pi(\varphi)$ has two elements $\{\pi_{\text{hol}}, \pi_{\text{gen}}\}$ where π_{gen} is generic and π_{hol} is holomorphic. One has moreover

$$C_\varphi = S_\varphi \simeq (\mathbb{Z}/2\mathbb{Z})^2$$

in the notation of §10.5, and the two Shelstad characters of C_φ associated to the elements of $\Pi(\varphi)$ are the ones which are trivial on the center Z of $\text{Sp}(4, \mathbb{C})$. Of course $\tau(\pi_{\text{gen}}) = 1$ and so $\tau(\pi_{\text{hol}})$ is the unique non-trivial character of C_φ which is trivial on the center $Z = \{\pm 1\}$ of $\text{Sp}(4, \mathbb{C})$.

Fix a $\psi \in \Psi_{\text{alg}}(\text{PGSp}(4))$ whose infinitesimal character has the eigenvalues $\pm w, \pm v$. One has to determine if $\Pi_\infty(\psi)$ contains the holomorphic discrete series and, if it is so, to determine the multiplicity of the unique $\pi \in \Pi(\psi)$ such that π_∞ is this holomorphic discrete series. Such a π is necessarily cuspidal as π_∞ is tempered, by a result of Wallach [W, Thm. 4.3] (as pointed out to us by Wallach, this discrete series case is actually significantly simpler than the general case treated there). We proceed by a case by case argument depending on the global Arthur parameter ψ :

Case (i) : (stable tempered case) $\psi = \pi_1$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(4))$. In this case ψ_∞ is a discrete series Langlands parameter. It follows from Arthur's multiplicity formula that $m(\pi) = 1$, as $C_\psi = Z$. The number of such π is the number $S(w, v)$ that we want to compute.

Case (ii) : $\psi = [4]$. The unique $\pi \in \Pi_{\text{disc}}(\text{PGSp}(4))$ with $\psi(\pi) = \psi$ is the trivial representation, for which π_∞ is not a discrete series.

Case (iii) : $\psi = \pi_1 \oplus \pi_2$ where $\pi_1, \pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$ and π_1, π_2 have different Hodge numbers. In this case one has

$$C_\psi = C_{\psi_\infty} = (\mathbb{Z}/2\mathbb{Z})^2.$$

Moreover, $r_\psi(\text{SL}(2, \mathbb{C})) = 1$ so ε_ψ is trivial and ψ_∞ is a discrete series parameter for $\text{PGSp}_4(\mathbb{R})$. If $\pi \in \Pi(\psi)$ is the unique element such that π_∞ is holomorphic, Arthur's multiplicity formula thus shows that $m(\pi) = 0$ as ε_ψ is trivial but $\tau(\pi_\infty)$ is not.

Case (iv) : $\psi = \pi_1 \oplus [2]$ where $\pi_1 \in \Pi_{\text{alg}}(\text{PGL}(2))$ with Hodge number $w \neq 1$ (which is actually automatic as $S(1) = 0$). Again one has

$$C_\psi = C_{\psi_\infty} = (\mathbb{Z}/2\mathbb{Z})^2.$$

This time $r_\psi(\text{SL}(2, \mathbb{C})) \neq 1$, and if s is the generator of C_ψ/Z , then

$$\varepsilon_\psi(s) = \varepsilon(\pi_1 \times 1) = \varepsilon(\pi_1) = (-1)^{(w+1)/2}.$$

The Adams-Johnson parameter ψ_∞ has an associated complex Levi subgroup L isomorphic to $\text{SO}(2, \mathbb{C}) \times \text{SO}(3, \mathbb{C})$ (see §10.2 and §10.5). It follows that the set $\Pi_\infty(\psi)$, which has two elements, contains the holomorphic discrete series (associated to the order 2 element in the center of L). For more details, see Chapter 9 where the general case $\text{Sp}(2g, \mathbb{R})$ will be studied. The character of this holomorphic discrete series relative to this ψ_∞ is again the non-trivial character of C_ψ trivial on Z by the discrete series case recalled above and Lemma 10.9. It follows that if $\pi \in \Pi(\psi)$ is the unique element such that $\pi_\infty = \pi_{\text{hol}}$, then by Arthur's multiplicity formula we have $m(\pi) = 0$ if $w \equiv 3 \pmod{4}$, and $m(\pi) = 1$ if $w \equiv 1 \pmod{4}$.

This concludes the proof of the proposition. \square

Remark 4.4. By the formula for $S(w)$, the first w for which a π as in case (iv) exists is for $w = 17$, for which $\psi(\pi) = \Delta_{17} \oplus [2]$. The representations π occurring in case (iv) have a long history, their existence had been conjectured by Saito and Kurokawa in 1977, and proved independently of this theory by Maass, Andrianov and Zagier. We refer to Arthur's paper [A2] for a discussion about this (and most of the discussion of this paragraph).

When $S(w, v) = 1$ we shall denote by $\Delta_{w,v}$ the unique element of $\pi \in \Pi_{w,v}(\text{PGSp}(4))$ such that $\psi(\pi) \in \Pi_{\text{cusp}}(\text{PGL}(4))$. Recall that an explicit formula for $S_{w,v}(\text{Sp}(4, \mathbb{Z}))$ has been computed by T. Tsushima [T]. See Table 6 for a sample of values. For $w < 25$, one observes that $S(w, v)$ is either 0 or 1. For those $w < 25$, there are exactly 7 forms $\Delta_{w,v}$, for the following values (w, v) :

$$(19, 7), (21, 5), (21, 9), (21, 13), (23, 7), (23, 9), (23, 13).$$

Contrary to the $\text{GL}(2)$ case where one has simple formulas for the $c_p(\Delta_w)$ thanks to the q -expansion of Eisenstein series or the product formula for Ramanujan's Δ_{11} , much less seems to be known at the moment for the $c_p(\pi)$ where $\pi \in \Pi_{w,v}(\text{PGSp}(4))$, even (say) for $\pi = \Delta_{w,v}$ and (w, v) in the list above. We refer to the recent work [RRST] for a survey on this important problem, as well as some implementation on SAGE.

To cite a few results especially relevant to our purposes here, let us mention first the work of Skoruppa [Sk] computing $c_p(\pi)$ for the first 22 primes p when π is any of the 18 elements in the $\Pi_{w,1}(\mathrm{PGSp}(4))$ for $w \leq 61$. Moreover, works of Faber and Van der Geer (see [VdG, §24, §25]) compute the trace of $c_p(\Delta_{v,w})$ in the standard 4-dimensional representations when $p \leq 37$, and even $c_p(\Delta_{w,v})$ itself when $p \leq 7$, whenever (w, v) is in the list above. In the work [ChLa] of the first author and Lannes, the first 4 of these forms, namely $\Delta_{19,7}$, $\Delta_{21,5}$, $\Delta_{21,9}$ and $\Delta_{21,13}$, appeared in the study of the Kneser p -neighbours of the Niemeier lattices. Properties of the Leech lattice also allowed those authors to compute $\mathrm{Trace}(c_p(\Delta_{w,v}))$ for those 4 pairs (w, v) up to $p \leq 79$.

4.5. An elementary lifting result for isogenies. Consider $\iota : G \rightarrow G'$ a central isogeny between semisimple Chevalley groups over \mathbb{Z} . The morphism ι is thus a finite flat group scheme homomorphism, $Z = \mathrm{Ker} \iota \subset Z(G)$ is a central multiplicative \mathbb{Z} -group scheme and $G' = G/Z$. The following proposition is easy to observe for all the isogenies we shall consider later, but it is perhaps more satisfactory to give a general proof.

Proposition 4.6. *ι induces a homeomorphism $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}}) \xrightarrow{\sim} G'(\mathbb{Q}) \backslash G'(\mathbb{A}) / G'(\widehat{\mathbb{Z}})$.*

Proof — By Prop. 3.5, it is enough to check that the map

$$G(\mathbb{Z}) \backslash G(\mathbb{R}) \rightarrow G'(\mathbb{Z}) \backslash G'(\mathbb{R})$$

induced by ι is a homeomorphism. As this map is continuous and open it is enough to show it is bijective. As the source and target are connected by Prop. 3.5, it is surjective. Moreover, it is injective if and only if the inverse image of $G'(\mathbb{Z})$ in $G(\mathbb{R})$ coincides with $G(\mathbb{Z})$, what we check now. The fppf exact sequence defined by ι leads to the following commutative diagram :

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z(\mathbb{R}) & \longrightarrow & G(\mathbb{R}) & \longrightarrow & G'(\mathbb{R}) \longrightarrow H^1(\mathbb{R}, Z) \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & Z(\mathbb{Z}) & \longrightarrow & G(\mathbb{Z}) & \longrightarrow & G'(\mathbb{Z}) \longrightarrow H^1(\mathbb{Z}, Z) \end{array}$$

The left vertical map is an isomorphism by Prop. 3.5. The right vertical one is an isomorphism as well, as so are the natural maps

$$\mathbb{Z}^* / (\mathbb{Z}^*)^n = H^1(\mathbb{Z}, \mu_n) \rightarrow H^1(\mathbb{R}, \mu_n) = \mathbb{R}^* / (\mathbb{R}^*)^n$$

for each integer $n \geq 1$. A simple diagram chasing concludes the proof. \square

Denote by $\iota^\vee : \widehat{G'} \rightarrow \widehat{G}$ the isogeny dual to ι . We now define a map

$$\mathcal{R}_\iota : \Pi(G') \longrightarrow \mathcal{P}(\Pi(G))$$

associated to ι as follows. If $\pi' = \pi'_\infty \otimes \pi'_f \in \Pi(G')$ we define $\mathcal{R}_\iota(\pi')$ as the set of representations $\pi \in \Pi(G)$ such that :

- (i) For each prime p the Satake parameter of π_p is $\iota^\vee(c_p(\pi))$,
- (ii) π_∞ is a constituent of the restriction to $G(\mathbb{R}) \rightarrow G'(\mathbb{R})$ of π'_∞ .

Observe that if $\pi \in \mathcal{R}_\iota(\pi')$ then π_p is uniquely determined by (i), moreover the restriction of π_∞ to $G(\mathbb{R})$ is a direct sum of finitely many representations, in particular $\mathcal{R}_\iota(\pi')$ is finite. We denote by $[\pi_\infty : \pi'_\infty]$ the multiplicity of π_∞ in $(\pi'_\infty)_{|G(\mathbb{R})}$.

If $\pi \in \Pi(H)$ we also denote by $m_H(\pi)$ for $m(\pi)$ to emphasize the \mathbb{Z} -group H .

Proposition 4.7. *If $\pi \in \Pi(G)$ then*

$$m_G(\pi) = \sum_{\pi' \in \Pi(G'), \pi \in \mathcal{R}_\iota(\pi')} m_{G'}(\pi') [\pi_\infty, \pi'_\infty].$$

In particular, for any $\pi \in \Pi_{\text{disc}}(G)$ there exists $\pi' \in \Pi_{\text{disc}}(G')$ such that $\pi \in \mathcal{R}_\iota(\pi')$, and for any $\pi' \in \Pi_{\text{disc}}(G')$ then $\mathcal{R}_\iota(\pi') \subset \Pi_{\text{disc}}(G)$.

Before giving the proof we need to recall certain properties of the Satake isomorphism. Following Satake, consider the \mathbb{C} -linear map

$$\iota^* : \mathcal{H}(G) \rightarrow \mathcal{H}(G')$$

sending the characteristic function of $G(\widehat{\mathbb{Z}})gG(\widehat{\mathbb{Z}})$ to the one of $G'(\widehat{\mathbb{Z}})\iota(g)G'(\widehat{\mathbb{Z}})$. It follows from [Sa, Prop. 7.1], that ι^* is a ring homomorphism. Indeed, it is enough to check the assumptions there. Let ι_p be the morphism $G(\mathbb{Q}_p) \rightarrow G'(\mathbb{Q}_p)$ induced by ι . Then $\iota_p(G(\mathbb{Q}_p))$ is a normal open subgroup of $G'(\mathbb{Q}_p)$. Moreover $\iota_p^{-1}(G'(\mathbb{Z}_p)) = G(\mathbb{Z}_p)$ as this latter group is a maximal compact subgroup of $G(\mathbb{Q}_p)$ (Tits) and ι_p is proper. Last but not least, the Cartan decomposition shows that ι_p induces an injection $G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \rightarrow G'(\mathbb{Z}_p) \backslash G'(\mathbb{Q}_p) / G'(\mathbb{Z}_p)$.

If V is a representation of $G'(\mathbb{A}_f)$, it defines by restriction by ι a representation V_ι of $G(\mathbb{A}_f)$ as well, and $V^{G(\widehat{\mathbb{Z}})} \subset V_\iota^{G(\widehat{\mathbb{Z}})}$.

Lemma 4.8. *Let V be a complex representation of $G'(\mathbb{A}_f)$ and let $T \in \mathcal{H}(G)$. The diagram*

$$\begin{array}{ccc} V^{G'(\mathbb{A}_f)} & \hookrightarrow & V_\iota^{G(\mathbb{A}_f)} \\ \downarrow \iota^*(T) & & \downarrow T \\ V^{G'(\mathbb{A}_f)} & \hookrightarrow & V_\iota^{G(\mathbb{A}_f)} \end{array}$$

is commutative.

Proof — We claim that if $\psi : G'(\mathbb{A}_f) \rightarrow \mathbb{C}$ is a locally constant function which is right $G'(\widehat{\mathbb{Z}})$ -invariant and with support in $\iota(G(\mathbb{A}_f))G'(\widehat{\mathbb{Z}})$, then

$$\int_{G'(\mathbb{A}_f)} \psi(g) dg = \int_{G(\mathbb{A}_f)} \psi(\iota(h)) dh.$$

Here the Haar measures dg and dh on $G'(\mathbb{A}_f)$ and $G(\mathbb{A}_f)$ are normalized so that $G'(\widehat{\mathbb{Z}})$ and $G(\widehat{\mathbb{Z}})$ have respective measure 1. The claim follows at once from $\iota^{-1}(G'(\widehat{\mathbb{Z}})) = G(\widehat{\mathbb{Z}})$ and $\iota(G(\mathbb{A}_f)) \subseteq G'(\mathbb{A}_f)$. \square

A finer property of ι^* is that it commutes with the Satake isomorphism. Recall that if \mathcal{H}_p denotes the Hecke algebra of $(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$, the Satake isomorphism is a canonical isomorphism

$$S_{G/\mathbb{Z}_p} : \mathcal{H}_p(G) \rightarrow R(\widehat{G})$$

where $R(\widehat{G})$ denotes the \mathbb{C} -algebra of polynomial class functions on \widehat{G} . Satake shows *loc. cit.* that the diagram

$$\begin{array}{ccc} \mathcal{H}_p(G) & \xrightarrow{S_{G/\mathbb{Z}_p}} & R(\widehat{G}) \\ \iota^* \downarrow & & \downarrow \iota^\vee \\ \mathcal{H}_p(G') & \xrightarrow{S_{G'/\mathbb{Z}_p}} & R(\widehat{G'}) \end{array}$$

is commutative, where $\iota^\vee : R(\widehat{G}) \rightarrow R(\widehat{G'})$ also denotes the restriction by ι^\vee .

Proof — Proposition 4.6 ensures that $f(g) \mapsto f(\iota(g))$ defines an isomorphism

$$\text{Res}_\iota : \mathcal{L}(G') \xrightarrow{\sim} \mathcal{L}(G).$$

The homomorphism ι^* defines a natural $\mathcal{H}(G)$ -module structure on $\mathcal{L}(G')$ and Lemma 4.8 ensures that Res is $\mathcal{H}(G)$ -equivariant for this structure on the left-hand side and the natural structure on the right-hand side. This isomorphism is obvious $G(\mathbb{R})$ -equivariant as well. As $\iota(G(\mathbb{R}))$ is open of finite index in $G'(\mathbb{R})$ we may replace the two \mathcal{L} 's above by $\mathcal{L}_{\text{disc}}$. The proposition follows then from (5). \square

Corollary 4.9. *Assume that $m_G(\pi) = 1$ for each $\pi \in \Pi_{\text{disc}}(G)$. Then $m_{G'}(\pi') = 1$ for each $\pi' \in \Pi_{\text{disc}}(G')$ as well. Moreover, the $\mathcal{R}_\iota(\pi')$ with $\pi' \in \Pi_{\text{disc}}(G')$ form a partition of $\Pi_{\text{disc}}(G)$. In other words, there is a surjective map*

$$f_\iota : \Pi_{\text{disc}}(G) \rightarrow \Pi_{\text{disc}}(G')$$

such that $f_\iota^{-1}(\{\pi'\}) = \mathcal{R}_\iota(\pi')$ for each $\pi' \in \Pi_{\text{disc}}(G')$.

This corollary applies for instance to the case $G = \text{Sp}(2g)$ for any $g \geq 1$ by Arthur's multiplicity formula (the case $g = 1$ being due to Labesse and Langlands).

Corollary 4.10. *If $G = \text{SO}(2, 2)$ then $m_G(\pi) = 1$ for any $\pi \in \Pi_{\text{disc}}(G)$.*

Proof — We just recalled that $m_H(\pi) = 1$ for any $\pi \in \Pi_{\text{disc}}(H)$ when $H = \text{SL}(2)$, hence for $H = \text{SL}(2) \times \text{SL}(2)$ as well. To conclude we apply Cor. 4.9 to the central isogeny

$$\text{SO}(2, 2)_{\text{sc}} \simeq \text{SL}(2) \times \text{SL}(2) \rightarrow \text{SO}(2, 2).$$

\square

4.11. **Symmetric square functoriality and $\Pi_{\text{cusp}}^\perp(\text{PGL}(3))$.** It follows from Theorem 3.9 that

$$\Pi_{\text{cusp}}^o(\text{PGL}(3)) = \Pi_{\text{cusp}}^\perp(\text{PGL}(3)).$$

Recall the \mathbb{C} -morphism $\text{Sym}^2 : \text{SL}(2, \mathbb{C}) \rightarrow \text{SL}(3, \mathbb{C})$. The following result is essentially due to Gelbart and Jacquet [GJ].

Proposition 4.12. *There is a unique bijection $\text{Sym}^2 : \Pi_{\text{cusp}}(\text{PGL}(2)) \rightarrow \Pi_{\text{cusp}}^o(\text{PGL}(3))$ such that for each $\pi \in \Pi_{\text{cusp}}(\text{PGL}(2))$ we have⁸*

$$c(\text{Sym}^2 \pi) = \text{Sym}^2 c(\pi).$$

It induces a bijection $\Pi_{\text{alg}}(\text{PGL}(2)) \xrightarrow{\sim} \Pi_{\text{alg}}^o(\text{PGL}(3))$.

If $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$ has Hodge number w , it follows that $\text{Sym}^2(\pi)$ has Hodge number $2w$. The proposition implies thus part (i) of Thm. 1.8. Observe in particular that the Hodge number of any $\pi \in \Pi_{\text{alg}}^o(\text{PGL}(3))$ is $\equiv 2 \pmod{4}$, as asserted in general by Prop. 1.7.

Let us deduce the proposition from Arthur's results. The uniqueness follows from the strong multiplicity one property for $\text{PGL}(3)$. Consider the \mathbb{Z} -group $\text{SL}(2) = \text{Sp}(2)$. If $\pi \in \Pi_{\text{disc}}(\text{SL}(2))$, Arthur associates to π an Arthur parameter which is either $[3]$ or belongs to $\Pi_{\text{cusp}}^o(3)$. But $\underline{\psi}(\pi) = [3]$ if and only if π is the trivial representation of $\text{SL}(2)$, and the trivial representation is well-known to be the only non cuspidal element in $\Pi_{\text{disc}}(\text{SL}(2))$. It follows that Arthur's parameterization furnishes a canonical map

$$\underline{\psi} : \Pi_{\text{cusp}}(\text{SL}(2)) \longrightarrow \Pi_{\text{cusp}}^o(\text{PGL}(3)),$$

which induces the standard representation $\text{SO}(3, \mathbb{C}) \subset \text{SL}(3, \mathbb{C})$ at the level of Langlands parameters. Arthur's multiplicity formula ensures that $\underline{\psi}$ is surjective (because C_ψ is trivial in those cases). It is essentially injective : if $\underline{\psi}(\pi) = \underline{\psi}(\pi')$ then $\pi_p = \pi'_p$ for each prime p and π_∞, π'_∞ belong to the same L -packet of $\text{SL}(2, \mathbb{R})$, i.e. are conjugate under $\text{PGL}(2, \mathbb{R})$.

On the other hand, Cor. 4.9 defines a natural surjective map

$$f : \Pi_{\text{disc}}(\text{SL}(2)) \rightarrow \Pi_{\text{disc}}(\text{PGL}(2))$$

such that for each $\pi \in \Pi_{\text{disc}}(\text{SL}(2))$, $c(\pi)$ is the image of $c(f(\pi))$ under the natural isogeny $\text{SL}(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C})$. Moreover, if $f(\pi) = f(\pi')$ then $\pi_p = \pi'_p$ for each prime p and π_∞, π'_∞ are conjugate under $\text{PGL}(2, \mathbb{R})$. If 1 is the trivial representation of $\text{PGL}(2)$, $f^{-1}(1)$ is the singleton made of the trivial representation of $\text{SL}(2)$. There is thus a unique bijection

$$\text{Sym}^2 : \Pi_{\text{cusp}}(\text{PGL}(2)) \rightarrow \Pi_{\text{cusp}}^o(\text{PGL}(3))$$

such that $\text{Sym}^2 \circ f = \underline{\psi}$. We conclude as under the isomorphism $\text{PGL}(2, \mathbb{C}) \xrightarrow{\sim} \text{SO}(3, \mathbb{C})$, the isogeny above coincides with the representation Sym^2 .

⁸We also have $L(\text{Sym}^2(\pi)_\infty) = \text{Sym}^2 L(\pi_\infty)$.

4.13. Tensor product functoriality and $\Pi_{\text{cusp}}^o(\text{PGL}(4))$. We consider the natural map

$$\mathcal{X}(\text{SL}(2, \mathbb{C})) \times \mathcal{X}(\text{SL}(2, \mathbb{C})) \rightarrow \mathcal{X}(\text{SL}(4, \mathbb{C}))$$

given by the tensor product $(x, y) \mapsto x \otimes y$ of conjugacy classes. Denote by

$$\Sigma_2(\Pi_{\text{cusp}}(\text{PGL}(2)))$$

the set of subsets of $\Pi_{\text{cusp}}(\text{PGL}(2))$ with two elements.

Proposition* 4.14. *There is a unique bijection $\Sigma_2(\Pi_{\text{cusp}}(\text{PGL}(2))) \xrightarrow{\sim} \Pi_{\text{cusp}}^o(\text{PGL}(4))$, that we shall denote*

$$\{\pi, \pi'\} \mapsto \pi \otimes \pi',$$

such that for each $\pi \neq \pi' \in \Pi_{\text{cusp}}(\text{PGL}(2))$, $c(\pi \otimes \pi') = c(\pi) \otimes c(\pi')$. It induces a bijection between $\Pi_{\text{alg}}^o(\text{PGL}(4))$ and the set of pairs $\{\pi, \pi'\}$ with $\pi \neq \pi'$ in $\Pi_{\text{alg}}(\text{PGL}(2))$.

If $\pi, \pi' \in \Pi_{\text{alg}}(\text{PGL}(2))$ have respective Hodge numbers $w \geq w'$, it follows that $\pi \otimes \pi'$ has Hodge numbers $w + w', w - w'$. The proposition implies thus part (ii) of Thm. 1.8 and fits Prop. 1.7.

To prove the proposition we start from the semisimple \mathbb{Z} -group $\text{SO}(2, 2)$. Its adjoint group is $\text{PGL}(2) \times \text{PGL}(2)$, thus Cor. 4.9 and Cor. 4.10 give a surjective map

$$f : \Pi_{\text{disc}}(\text{SO}(2, 2)) \rightarrow \Pi_{\text{disc}}(\text{PGL}(2) \times \text{PGL}(2)) = \Pi_{\text{disc}}(\text{PGL}(2)) \times \Pi_{\text{disc}}(\text{PGL}(2))$$

such that for each $\rho \in \Pi_{\text{cusp}}(\text{SO}(2, 2))$, $c(\rho)$ is the image of $c(f(\rho))$ under the natural "tensor-product" isogeny $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow \text{SO}(4, \mathbb{C})$. In particular, if $f(\rho) = f(\rho')$ then $\psi(\rho) = \psi(\rho')$ and it thus makes sense to consider, for any two given $\pi, \pi' \in \Pi_{\text{disc}}(\text{PGL}(2))$, the global Arthur parameter

$$\psi(\pi, \pi') = \psi(\rho) \in \Psi_{\text{glob}}(\text{SO}(2, 2))$$

of any element ρ in $\Pi_{\text{disc}}(\text{SO}(2, 2))$ such that $f(\rho) = (\pi, \pi')$. It is clear that $\psi(\pi, \pi') = \psi(\pi', \pi)$.

Proposition* 4.15. *Let $\pi, \pi' \in \Pi_{\text{disc}}(\text{PGL}(2))$.*

- (i) *If π, π' are both the trivial representation then $\psi(\pi, \pi') = [3] \oplus [1]$,*
- (ii) *If π' is the trivial representation and π is cuspidal then $\psi(\pi, \pi') = \pi[2]$,*
- (iii) *If $\pi = \pi'$ is cuspidal, then $\psi(\pi, \pi') = \text{Sym}^2 \pi \oplus [1]$,*
- (iv) *If π, π' are distinct and cuspidal, then $\psi(\pi, \pi') \in \Pi_{\text{cusp}}^o(\text{PGL}(4))$. Moreover, $\psi(\pi, \pi')$ determines the pair $\{\pi, \pi'\}$.*

Note that assertion (iii) makes sense by Proposition 4.12.

Proof — Assertions (i), (ii) and (iii) follow from an immediate inspection of Satake parameters. We will now prove (iv) using Arthur's multiplicity formula.

Fix distinct $\pi, \pi' \in \Pi_{\text{cusp}}^o(\text{PGL}(2))$ as well as $\rho \in \Pi_{\text{disc}}(\text{SO}(2, 2))$ such that $f(\rho) = (\pi, \pi')$. Recall the outer automorphism s of $\text{SO}(2, 2)$ defined by an orthogonal reflexion as in Remark 3.2. This automorphism factors through an outer automorphism of $\text{PGL}(2) \times \text{PGL}(2)$ (i.e. exchanging the two factors), in particular $s(\rho) \neq \rho$.

Consider now any $\pi'', \pi''' \in \Pi_{\text{disc}}(\text{PGL}(2))$ such that $\psi(\pi, \pi') = \psi(\pi'', \pi''')$. Choose $\rho' \in \Pi_{\text{disc}}(\text{SO}(2, 2))$ such that $f(\rho') = (\pi'', \pi''')$. As the Arthur parameters appearing in (i), (ii) and (iii) are all the possible non-cuspidal global Arthur parameters for $\text{SO}(2, 2)$, it only remains to show that under those conditions we have $\{\pi, \pi'\} = \{\pi'', \pi'''\}$.

The representation ρ_∞ is generic as $f(\rho)$ is, thus the global Arthur parameters of ρ and ρ' have no $\text{SL}(2, \mathbb{C})$ -factor. The assumption $\psi(\rho) = \psi(\rho')$ implies then at the infinite place that up to replacing (π'', π''') by (π''', π'') and ρ' by $\tau(\rho')$, we may assume that ρ'_∞ and ρ_∞ are in the same L -packet of $\text{SO}(2, 2)(\mathbb{R})$, i.e. in the same $\text{PGL}(2, \mathbb{R}) \times \text{PGL}(2, \mathbb{R})$ -orbit. In this case we may even choose ρ' so that $\rho'_\infty = \rho_\infty$. By assumption, $\rho, \rho' \in \Pi(\psi)$ where $\psi = \pi \otimes \pi'$. Arthur's multiplicity formula shows that the set $\{\rho, \rho', s(\rho), s(\rho')\}$ has ≤ 2 elements. As $s(\rho) \neq \rho$ it follows that $\rho' \in \{\rho, s(\rho)\}$, hence $\{\pi, \pi'\} = \{\pi'', \pi'''\}$. \square

Let us finally prove Proposition 4.14. If $\pi, \pi' \in \Pi_{\text{cusp}}(\text{PGL}(2))$ are distinct we set

$$\pi \otimes \pi' = \psi(\pi, \pi'),$$

so that $c(\pi \otimes \pi') = c(\pi) \otimes c(\pi')$ by definition, and also $L((\pi \otimes \pi')_\infty) = L(\pi_\infty) \otimes L(\pi'_\infty)$. It follows from Prop. 4.15 that $\{\pi, \pi'\} \mapsto \pi \otimes \pi'$ defines an injection $\Sigma_2(\Pi_{\text{cusp}}(\text{PGL}(2))) \rightarrow \Pi_{\text{cusp}}^o(\text{PGL}(4))$ which is actually surjective by Theorem 3.9. The last statement is a simple exercise using the relation $L((\pi \otimes \pi')_\infty) = L(\pi_\infty) \otimes L(\pi'_\infty)$.

4.16. Λ^* functorality and $\Pi_{\text{cusp}}^o(\text{PGL}(5))$. We denote by Λ^* the irreducible representation $\text{Sp}(4, \mathbb{C}) \rightarrow \text{SL}(5, \mathbb{C})$, so that $\Lambda^2 \mathbb{C}^4 = \Lambda^* \oplus 1$. If $\pi \in \Pi_{\text{cusp}}^s(\text{PGL}(4))$, there is a unique element $\widetilde{c(\pi)} \in \mathcal{X}(\text{Sp}(4, \mathbb{C}))$ such that $\text{St}(\widetilde{c(\pi)}) = c(\pi)$ (see § 3.8).

Proposition* 4.17. *There is a unique bijection*

$$\Pi_{\text{disc}}^s(\text{PGL}(4)) \xrightarrow{\sim} \Pi_{\text{disc}}^o(\text{PGL}(5)),$$

denoted by $\pi \mapsto \Lambda^* \pi$, such that for each $\pi \in \Pi_{\text{disc}}^s(\text{PGL}(4))$ we have

$$\Lambda^*(c(\pi)) = c(\Lambda^* \pi) \quad \text{and} \quad \Lambda^* \circ \widetilde{L}(\pi_\infty) = L((\Lambda^* \pi)_\infty).$$

It induces a bijection $\Pi_{\text{alg}}^s(\text{PGL}(4)) \xrightarrow{\sim} \Pi_{\text{alg}}^o(\text{PGL}(5))$.

The last assertion follows from the first one. If $\pi \in \Pi_{\text{alg}}^s(\text{PGL}(4))$ has Hodge numbers $w > v$, then $\Lambda^* \pi$ has Hodge numbers $w + v > w - v$. Let us prove now the first assertion. This will conclude as well the proof of Thm. 1.8.

The adjoint group of $\text{Sp}(4)$ is $\text{PGSp}(4) = \text{SO}(3, 2)$, thus Cor. 4.9 gives a natural surjective map

$$f : \Pi_{\text{disc}}(\text{Sp}(4)) \rightarrow \Pi_{\text{disc}}(\text{SO}(3, 2))$$

such that for each $\rho \in \Pi_{\text{cusp}}(\text{Sp}(4))$, $c(\rho)$ is the image of $c(f(\rho))$ under the natural Λ^* isogeny $\text{Sp}(4, \mathbb{C}) \rightarrow \text{SO}(5, \mathbb{C})$. Moreover, if $f(\rho) = f(\rho')$ then $\rho_p = \rho'_p$ for each prime p and $\rho_\infty, \rho'_\infty$ belong to the set of 1 or 2 constituents of the restriction to $\text{Sp}(4, \mathbb{R})$ of the representation $f(\rho)_\infty$. In particular, $\psi(\rho) = \psi(\rho')$ and it thus makes sense to consider the global Arthur parameter

$$\widetilde{\psi}(\pi) = \psi(\rho) \in \Psi_{\text{glob}}(\text{Sp}(4))$$

of any element π in $\Pi_{\text{disc}}(\text{SO}(3, 2))$.

Proposition* 4.18. *Let $\pi \in \Pi_{\text{disc}}(\text{SO}(3, 2))$.*

- (i) *If $\psi(\pi) = [4]$ then $\tilde{\psi}(\pi) = [5]$,*
- (ii) *If $\psi(\pi) = \pi_1 \oplus [2]$ with $\pi_1 \in \Pi_{\text{cusp}}(\text{PGL}(2))$ then $\tilde{\psi}(\pi) = \pi_1[2] \oplus [1]$,*
- (iii) *If $\psi(\pi) = \pi_1 \oplus \pi_2$ with distinct $\pi_1, \pi_2 \in \Pi_{\text{cusp}}(\text{PGL}(2))$ then $\tilde{\psi}(\pi) = \pi_1 \otimes \pi_2 \oplus [1]$,*
- (iv) *If $\psi(\pi) \in \Pi_{\text{cusp}}^s(\text{PGL}(4))$ then $\tilde{\psi}(\pi) \in \Pi_{\text{cusp}}^o(\text{PGL}(5))$. Moreover, $\tilde{\psi}(\pi)$ determines $\psi(\pi)$.*

Note that assertion (iii) makes sense by Prop. 4.14.

Proof — Assertions (i), (ii) and (iii) follow from an immediate inspection of Satake parameters. Let us check (iv). If $\pi \in \Pi_{\text{cusp}}^s(\text{PGL}(4))$, Theorem 3.9 gives us a $\pi' \in \Pi_{\text{disc}}(\text{SO}(3, 2))$ such that $\pi = \psi(\pi')$ and we may even assume that π'_∞ is generic by the multiplicity formula. Let $\rho \in \Pi_{\text{disc}}(\text{Sp}(4))$ such that $f(\rho) = \pi'$, then ρ_∞ is generic as well so the global Arthur parameter $\tilde{\psi}(\pi') = \psi(\rho)$ has no $\text{SL}(2, \mathbb{C})$ -factor.

Observe that the non-cuspidal global Arthur parameters for $\text{Sp}(4)$ are exactly the ones of the type (i), (ii) and (iii) above. Let us check that $\psi(\rho)$ is not as in case (iii). Let π_1, π_2 be distinct elements in $\Pi_{\text{cusp}}(\text{PGL}(2))$. By Arthur's multiplicity formula for $\text{SO}(3, 2)$ there is a unique element $\pi'' \in \Pi_{\text{disc}}(\text{SO}(3, 2))$ such that $\psi(\pi'') = \pi_1 \oplus \pi_2$ and π''_∞ is generic. Choose $\rho' \in \Pi_{\text{disc}}(\text{Sp}(4))$ such that $f(\rho') = \pi''$. Of course,

$$f(\rho') = \pi'' \neq \pi' = f(\rho)$$

as $\psi(\pi') \neq \psi(\pi'')$. Assume now that $\psi(\rho) = \psi(\rho')$. This implies that ρ and ρ' are in the same L -packet at each place. But ρ_∞ and ρ'_∞ are both generic, they are thus conjugate under $\text{PGSp}(4, \mathbb{R})$. We may thus assume that $\rho = \rho'$, which is absurd as $f(\rho) \neq f(\rho')$. One checks similarly that $\tilde{\psi}(\pi)$ determines π . \square

If $\pi \in \Pi_{\text{cusp}}^s(\text{PGL}(4))$, Theorem 3.9 gives us a $\rho \in \Pi_{\text{disc}}(\text{SO}(3, 2))$ such that $\pi = \psi(\rho)$. Set

$$\Lambda^* \pi = \tilde{\psi}(\rho).$$

It belongs to $\Pi_{\text{cusp}}^o(\text{PGL}(5))$ by the previous lemma and does not depend on the choice of ρ . \square

4.19. The Langlands group of \mathbb{Z} and Sato-Tate groups. Let us now define the conjectural *motivic Langlands group of \mathbb{Z}* , which is a certain quotient of the Langlands group of \mathbb{Q} , for which we shall adopt Kottwitz' presentation in the last section of [K1].

Denote by $W_{\mathbb{R}}^{\text{mot}}$ the quotient of $W_{\mathbb{R}}$ by the identity component of its center, namely $\mathbb{R}_{>0}$, and define $\Pi_{\text{mot}}(\text{PGL}(n))$ as the subset of $\pi \in \Pi_{\text{cusp}}(\text{PGL}(n))$ such that $L(\pi_\infty)$ factors through $W_{\mathbb{R}}^{\text{mot}}$. Of course, $\Pi_{\text{alg}}^\perp(\text{PGL}(n)) \subset \Pi_{\text{mot}}(\text{PGL}(n))$. The conjectural group $\mathcal{L}_{\mathbb{Z}}$ is by definition the quotient of the conjectural Langlands group of \mathbb{Q} whose irreducible n -dimensional complex representations parameterize the elements of $\Pi_{\text{mot}}(\text{PGL}(n))$ for each $n \geq 1$.

In particular it is a perfect and compact topological group equipped with :

- (a) a conjugacy class $\text{Frob}_p \subset \mathcal{L}_{\mathbb{Z}}$ for each prime p ,
- (b) a conjugacy class of continuous group homomorphisms $\iota : W_{\mathbb{R}}^{\text{mot}} \rightarrow \mathcal{L}_{\mathbb{Z}}$,

and which satisfies the following two axioms (i) and (ii). Recall the parameterization map $c : \Pi(\text{PGL}(n)) \rightarrow \mathcal{X}(\text{SL}(n, \mathbb{C}))$ defined in §3.7. Denote by $\text{Irr}_n(\mathbb{Z})$ the set of isomorphism classes of irreducible continuous representations $r : \mathcal{L}_{\mathbb{Z}} \rightarrow \text{GL}(n, \mathbb{C})$ and define a map $c' : \text{Irr}_n(\mathbb{Z}) \rightarrow \mathcal{X}(\text{SL}(n, \mathbb{C}))$, associating to any r the collection of semisimple conjugacy classes $r(\text{Frob}_p)$ and the infinitesimal character of the Langlands parameter $r|_{\iota}$.

- (i) The Frob_p are equi-distributed in the space of conjugacy classes of $\mathcal{L}_{\mathbb{Z}}$ equipped with its invariant $\mathcal{L}_{\mathbb{Z}}$ -measure of mass 1.
- (ii) For any $n \geq 1$, $c'(\text{Irr}_n(\mathbb{Z})) = c(\Pi_{\text{mot}}(\text{PGL}(n)))$.

Observe in particular that the union of the conjugacy classes Frob_p is dense in $\mathcal{L}_{\mathbb{Z}}$, so that $r \mapsto c'(r)$ is injective.

A first property of $\mathcal{L}_{\mathbb{Z}}$ is that it is connected. Indeed, it follows from [M, §3] that for $n > 1$ there is no $\pi \in \Pi_{\text{mot}}(\text{PGL}(n))$ such that $L(\pi_{\infty})$ is trivial on $W_{\mathbb{C}}$. In particular, the (connected) conjugacy class of $\iota(W_{\mathbb{C}})$ generates a dense subgroup of $\mathcal{L}_{\mathbb{Z}}$. This is the automorphic representation counterpart of Minkowski's theorem asserting that $\text{Spec}(\mathbb{Z})$ is simply connected. Langlands also expects that the irreducible projective representations $\mathcal{L}_{\mathbb{Z}} \rightarrow \text{PGL}(n, \mathbb{C})$ are in one-to-one correspondence with the L -packets in $\Pi_{\text{cusp}}(\text{SL}(n))$. From this perspective, the results of § 4.5 assert in particular that $\mathcal{L}_{\mathbb{Z}}$ is simply connected.

If $\pi \in \Pi_{\text{mot}}(\text{PGL}(n))$, let $r(\pi) : \mathcal{L}_{\mathbb{Z}} \rightarrow \text{GL}(n, \mathbb{C})$ be the representation given by property (ii). Define the Langlands-Sato-Tate group of π as the compact connected Lie group

$$\mathcal{L}_{\pi} = \text{Im}(r(\pi)).$$

It is equipped with a tautological irreducible complex representation of dimension n . By property (i), the Satake parameters of the π_p are thus equidistributed for the invariant measure on the space of conjugacy classes of \mathcal{L}_{π} .

Our goal in this paragraph is to determine the possible Langlands-Sato-Tate groups of the π considered in this paper (assuming of course the existence of $\mathcal{L}_{\mathbb{Z}}$). If $\pi \in \Pi_{\text{alg}}^{\perp}(\text{PGL}(n))$ define \mathcal{A}_{π} as the compact symplectic group of rank $n/2$ if $s(\pi) = -1$, the compact special orthogonal group $\text{SO}(n, \mathbb{R})$ otherwise. As already seen, \mathcal{L}_{π} is isomorphic to a subgroup of \mathcal{A}_{π} .

Proposition 4.20. *Assume the existence of $\mathcal{L}_{\mathbb{Z}}$. Let $\pi \in \Pi_{\text{alg}}^{\perp}(\text{PGL}(n))$ and assume $n \leq 8$. Then $\mathcal{L}_{\pi} \simeq \mathcal{A}_{\pi}$ unless :*

- (i) $s(\pi) = (-1)^{n+1}$ and there exists a $\pi' \in \Pi_{\text{alg}}(\text{PGL}(2))$ such that $r(\pi) = \text{Sym}^{n-1} r(\pi')$. In this case $\mathcal{L}_{\pi} \simeq \text{SU}(2)$ if n is even, $\text{SO}(3, \mathbb{R})$ if n is odd.
- (ii) $n = 6$, $s(\pi) = -1$, and there exists two distinct $\pi', \pi'' \in \Pi_{\text{alg}}(\text{PGL}(2))$ such that $r(\pi) = r(\pi') \otimes \text{Sym}^2 r(\pi'')$. In this case $\mathcal{L}_{\pi} \simeq \text{SU}(2) \times \text{SO}(3, \mathbb{R})$.
- (iii) $n = 7$ and \mathcal{L}_{π} is the compact simple group of type G_2 .

- (iv) $n = 8$, $s(\pi) = 1$ and there exists $\pi' \in \Pi_{\text{alg}}(\text{PGL}(2))$, $\pi'' \in \Pi_{\text{alg}}^s(\text{PGSp}(4))$, such that $r(\pi) \simeq r(\pi') \otimes r(\pi'')$. In this case \mathcal{L}_π is the quotient of $\text{SU}(2) \times \text{Sp}(4)$ by the diagonal central $\{\pm 1\}$.
- (v) $n = 8$, $s(\pi) = 1$ and there exists two distinct $\pi', \pi'' \in \Pi_{\text{alg}}(\text{PGL}(2))$ such that $r(\pi) = r(\pi') \otimes \text{Sym}^3 r(\pi'')$. In this case \mathcal{L}_π is the quotient of $\text{SU}(2) \times \text{SU}(2)$ by the diagonal central $\{\pm 1\}$.
- (vi) $n = 8$, $s(\pi) = -1$ and there exists distinct $\pi', \pi'', \pi''' \in \Pi_{\text{alg}}(\text{PGL}(2))$ such that $r(\pi) = r(\pi') \otimes r(\pi'') \otimes r(\pi''')$. In this case \mathcal{L}_π is the quotient of $\text{SU}(2)^3$ by the central subgroup $\{(\epsilon_i) \in \{\pm 1\}^3, \epsilon_1 \epsilon_2 \epsilon_3 = 1\}$.

Indeed, the only simply connected quasi-simple compact Lie groups having a self-dual finite dimensional irreducible representation of dimension ≤ 8 are in types : A_1 in each dimension, $B_2 = C_2$ in dimensions 4 and 5, C_3 in dimension 6, G_2 and B_3 in dimension 7, and C_4 in dimension 8 (three representations permuted by triality).

5. $\Pi_{\text{disc}}(\text{SO}(7))$ AND $\Pi_{\text{alg}}^s(\text{PGL}(6))$

5.1. The semisimple \mathbb{Z} -group $\text{SO}(7)$. Consider the semisimple classical \mathbb{Z} -group

$$G = \text{SO}(7) = \text{SO}_{E_7},$$

i.e. the special orthogonal group of the root lattice E_7 equipped with its canonical positive definite integral quadratic form. Let $W(E_7)$ denote the Weyl group of the root system of E_7 , let $\varepsilon : W(E_7) \rightarrow \{\pm 1\}$ be the signature and $W(E_7)^+ = \text{Ker } \varepsilon$. As the Dynkin diagram of E_7 has no non-trivial automorphism one has $O(E_7) = W(E_7)$, thus

$$G(\mathbb{Z}) = W(E_7)^+.$$

The group $W(E_7)^+$ has order $1451520 = 7! \cdot 2^5 \cdot 3^2$, it is isomorphic via the reduction modulo 2 to the finite simple group $G(\mathbb{F}_2) \simeq \text{Sp}(6, \mathbb{F}_2)$ ([Bki, Ch. VI, Ex. 3 §4]).

The class set $\text{Cl}(G) \simeq X_7$ has one element as $X_7 = \{E_7\}$ (§ 3.1, § 3.4). By Arthur's multiplicity formula, each $\pi \in \Pi_{\text{disc}}(G)$ has multiplicity 1. It follows from Prop. 3.6 that the number $m(V)$ of $\pi \in \Pi_{\text{disc}}(G)$ such that π_∞ is a given irreducible representation of $G(\mathbb{R})$ is

$$m(V) = \dim V^{W(E_7)^+},$$

which is exactly the number computed in the first chapter § 2.5 Case I. We refer to Table 2 for a sample of results.

The dual group of $\text{SO}(7)$ is $\widehat{G} = \text{Sp}(6, \mathbb{C})$.

5.2. Parameterization by the infinitesimal character. From the point of view of Langlands parameterization, it is more natural to label the irreducible representations of $G(\mathbb{R})$ by their infinitesimal character rather than their highest weight.

Let H be a compact connected Lie group, fix $T \subset H$ a maximal torus and $\Phi^+ \subset X^*(T)$ a set of positive roots as in § 2.2. Denote by $\rho \in X^*(T)[1/2]$ the half sum of the elements of Φ^+ . Under the Harish-Chandra isomorphism, the infinitesimal character of the irreducible representation V_λ of H of highest weight λ is the $W(H, T)$ -orbit of $\lambda + \rho$.

For instance if $H = \text{SO}(n, \mathbb{R})$, and in terms of the standard root data defined in § 2.5,

$$\rho = \begin{cases} \frac{2l-1}{2}e_1 + \frac{2l-3}{2}e_2 + \cdots + \frac{1}{2}e_l & \text{if } n = 2l + 1, \\ (l-1)e_1 + (l-2)e_2 + \cdots + e_{l-1} & \text{if } n = 2l. \end{cases}$$

The map $\lambda \mapsto \lambda + \rho = \sum_i \frac{w_i}{2}e_i$ induces thus a bijection between the dominant weights and the collection of $w_1 > w_2 > \cdots > w_l$ where the w_i are odd positive integers when $n = 2l + 1$, even integers with $w_{l-1} > |w_l|$ when $n = 2l$.

Definition 5.3. Let $n \geq 1$ be an integer, set $l = [n/2]$, and let $\underline{w} = (w_1, \dots, w_l)$ where $w_1 > w_2 > \cdots > w_l \geq 0$ are distinct positive integers all congruent to n modulo 2. We denote by

$$U_{\underline{w}}$$

the finite dimensional irreducible representation V_λ of $\text{SO}(n, \mathbb{R})$ such that $\lambda + \rho = \sum_i \frac{w_i}{2}e_i$.

As an example, observe that if $H_m(\mathbb{R}^n)$ is the representation of $\mathrm{SO}(n, \mathbb{R})$ defined in § 2.7, then $H_m(\mathbb{R}^n) = U_{\underline{w}}$ for

$$w = \begin{cases} (2m + n - 2, n - 4, n - 6, \dots, 3, 1) & \text{if } n \equiv 1 \pmod{2}, \\ (2m + n, n - 2, n - 4, \dots, 2, 0) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

The infinitesimal character $\lambda + \rho$ is related to Langlands parameterization as follows. Assume to simplify that H is semisimple and that $-1 \in W(H, T)$. This is always the case if $H = G(\mathbb{R})$ and G is semisimple over \mathbb{Z} , and for $H = \mathrm{SO}(n, \mathbb{R})$ this holds if and only if $n \not\equiv 2 \pmod{4}$. Then the Langlands dual group of H is a connected semisimple complex group \widehat{H} . Recall that \widehat{H} is equipped with a maximal torus \widehat{T} , a set of positive roots $(\Phi^\vee)^+$ for $(\widehat{H}, \widehat{T})$, and an isomorphism between the dual based root datum of $(\widehat{H}, \widehat{T}, (\Phi^\vee)^+)$ and the one of (H, T, Φ^+) . In particular, $X_*(\widehat{T})$ and $X^*(T)$ are identified by definition. The Langlands parameter of V_λ is up to \widehat{H} -conjugation the unique continuous homomorphism $L(V_\lambda) : W_{\mathbb{R}} \rightarrow \widehat{H}$ with finite centralizer and such that in Langlands' notation

$$L(V_\lambda)(z) = (z/\bar{z})^{\lambda+\rho} \in \widehat{T} \quad \forall z \in \mathbb{C}^* = W_{\mathbb{C}}.$$

When $H = \mathrm{SO}(n, \mathbb{R})$ and $\underline{w} = (w_1, w_2, \dots, w_l)$ is as in definition 5.3, it follows that in the standard representation $\mathrm{St} : \widehat{H} \rightarrow \mathrm{GL}(2l, \mathbb{C})$ of the classical group \widehat{H} , we have

$$\mathrm{St} \circ L(U_{\underline{w}}) \simeq \bigoplus_{i=1}^l I_{w_i}.$$

This is the reason why the normalization above will be convenient.

Definition 5.4. Let G be the semisimple classical definite \mathbb{Z} -group $\mathrm{SO}(n)$. If $\underline{w} = (w_1, \dots, w_l)$ is as in Def. 5.3 we define

$$\Pi_{\underline{w}}(G) = \{\pi \in \Pi_{\mathrm{disc}}(G), \pi_\infty \xrightarrow{\sim} U_{\underline{w}}\}$$

and set $m(\underline{w}) = |\Pi_{\underline{w}}(G)|$.

If $\pi \in \Pi_{\mathrm{disc}}(G)$, we shall say that π has Hodge numbers \underline{w} if $\pi \in \Pi_{\underline{w}}(G)$.

5.5. Endoscopic partition of $\Pi_{\mathrm{disc}}(\mathrm{SO}(7))$. Recall that if $\pi \in \Pi_{\underline{w}}(\mathrm{SO}(n))$, it has a global Arthur parameter

$$\psi(\pi) = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(\mathrm{SO}(n))$$

whose equivalence class is well-defined. The associated collection $(k, (n_i), (d_i))$ will be called the *endoscopic type* of π . As for $\psi(\pi)$, the endoscopic type will be called *stable* if $k = 1$, and *tempered* if $d_i = 1$ for $i = 1, \dots, k$. By Lemma 3.16, $\psi(\pi)$ is stable and tempered if and only if it belongs to $\Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}(2l))$ where $l = [n/2]$.

So far we have computed $|\Pi_{\underline{w}}(\mathrm{SO}(7))|$ for any possible Hodge numbers \underline{w} . Our next aim will be to compute the number of elements in $\Pi_{\underline{w}}(\mathrm{SO}(7))$ of each possible endoscopic type. As we shall see, thanks to Arthur's multiplicity formula and our previous computation of $S(w)$, $S(w, v)$ and $O^*(w)$, we will be able to compute the contribution of each endoscopic type except one, namely the stable and tempered type, which is actually $S(w_1, w_2, w_3)$.

We will in turn obtain this later number from our computation of $|\Pi_w(\mathrm{SO}(7))|$. The Corollary 1.5 and Table 7 will follow from these computations.

Fix a triple $\underline{w} = (w_1, w_2, w_3)$. Fix as well once and for all a global Arthur parameter

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(\mathrm{SO}(7))$$

such that the semisimple conjugacy class $\mathrm{St}(z_{\psi_\infty})$ in $\mathfrak{sl}(6, \mathbb{C})$ has the eigenvalues

$$\pm w_1, \pm w_2, \pm w_3.$$

Let us denote by π the unique element in $\Pi(\psi)$. We shall make explicit Arthur's multiplicity formula for $m(\pi)$, which is either 0 or 1 as $m_\psi = 1$, following §3.23.1. Recall the important groups

$$C_\psi \subset C_{\psi_\infty} \subset \mathrm{Sp}(6, \mathbb{C}).$$

For each $1 \leq i \leq k$ one has a distinguished element $s_i \in C_\psi$ (§ 3.20). Those k -elements s_i generate $C_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^k$ and their product generates the center $Z = \{\pm 1\}$ of $\mathrm{Sp}(6, \mathbb{C})$.

5.5.1. The stable case. This is the case $k = 1$, i.e. $C_\psi = Z$, for which the multiplicity formula trivially gives $m(\pi) = 1$. Let us describe the different possibilities for ψ . One has $\psi(\pi) = \pi_1[d_1]$ with $d_1|6$, $\pi_1 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}(6/d_1))$ and $(-1)^{d_1-1}s(\pi_1) = -1$.

Case (i): $\psi = \pi_1$ where $\pi_1 \in \Pi_{\mathrm{alg}}^s(\mathrm{PGL}(6))$, this is the unknown we want to compute.

Case (ii) : $\psi = \pi_1[2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(3))$, say of Hodge number $u > 2$ (so $u \equiv 2 \pmod{4}$). This occurs if and only if \underline{w} has the form $(u+1, u-1, 1)$. Recall that $\pi_1 = \mathrm{Sym}^2 \pi'$ for a unique $\pi' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}(2))$ with Hodge number $u/2$.

Case (iii) : $\psi = \pi_1[3]$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}(2))$, say of Hodge number $u > 1$ (an odd integer). This occurs if and only if w has the form $(u+2, u, u-2)$.

Case (iv) : $\psi = [6]$. This occurs if and only if $w = (5, 3, 1)$, and π is then the trivial representation of G .

5.5.2. Endoscopic cases of type $(n_1, n_2) = (4, 2)$. In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $C_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z . One will have to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\mathrm{Min}(d_1, d_2)}$ in each case. Recall that $\rho^\vee : C_{\psi_\infty} \rightarrow \{\pm 1\}$ is the fundamental character defined in § 3.23.1. There are three cases.

Case (v) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1 \in \Pi_{\mathrm{alg}}^s(\mathrm{PGL}(4))$ and $\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}(2))$. Denote by $a > b$ the Hodge numbers of π_1 and by c the Hodge number of π_2 . One has $\{a, b, c\} = \{w_1, w_2, w_3\}$. One sees that

$$\rho^\vee(s_1) = 1 \text{ iff } a > c > b$$

But $\varepsilon_\psi(s_1) = 1$ as all the d_i are 1 (tempered case). It follows that

$$m(\pi) = 1 \iff a > c > b$$

or which is the same iff $(w_1, w_2, w_3) = (a, c, b)$.

Case (vi) : $d_1 = 1, d_2 = 2$, i.e. $\psi = \pi_1 \oplus [2]$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(4))$ has Hodge numbers $w_1 > w_2$ with $w_2 > 1$. One sees that $\rho^\vee(s_1) = -1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_1) = (-1)^{(w_1+w_2+2)/2}$, it follows that

$$m(\pi) = 1 \Leftrightarrow w_1 + w_2 \equiv 0 \pmod{4}.$$

Case (vii) : $d_1 = 4, d_2 = 1$, i.e. $\psi = [4] \oplus \pi_2$ where $\pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$ has Hodge numbers w_1 with $w_1 > 3$. One sees that $\rho^\vee(s_1) = -1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_2) = (-1)^{(w_1+1)/2}$, it follows that

$$m(\pi) = 1 \Leftrightarrow w_1 \equiv 1 \pmod{4}.$$

5.5.3. Endoscopic cases of type $(n_1, n_2, n_3) = (2, 2, 2)$. In this case $k = 3$, and C_ψ is generated by Z and s_1, s_2 . There are two cases.

Case (viii) : (*tempered case*) $d_i = 1$ for each i , i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3$ where each $\pi_i \in \Pi_{\text{alg}}(\text{PGL}(2))$ and π_i has Hodge number w_i . and $\pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$. Of course ε_ψ is trivial here, so $m(\pi) = 1$ if and only if ρ^\vee is trivial on C_ψ . But $C_\psi = C_{\psi_\infty}$ and ρ^\vee is a non-trivial character, so

$$m(\pi) = 0$$

in all the cases.

Case (ix): $d_1 = d_2 = 1$ and $d_3 = 2$, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus [2]$ where $\pi_1, \pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$ have respective Hodge numbers $w_1 > w_2$, with $w_2 > 1$. One has thus $\rho^\vee(s_1) = -1$ and $\rho^\vee(s_2) = 1$. On the other hand for $i = 1, 2$ one has $\varepsilon_\psi(s_i) = \varepsilon(\pi_i) = (-1)^{(w_i+1)/2}$. It follows that

$$m(\pi) = 1 \Leftrightarrow (w_1, w_2) \equiv (1, 3) \pmod{4}.$$

5.6. Conclusions. First, one obtains the value of $S(w_1, w_2, w_3)$ as the difference between $m(w_1, w_2, w_3)$ and the sum of the eight last contributions above. For instance, one sees that if $w_1 - 2 > w_2 > w_3 + 2 > 3$ then

$$S(w_1, w_2, w_3) = m(w_1, w_2, w_3) - S(w_1, w_3) \cdot S(w_2).$$

It turns out that all the formulas for the nine cases considered above perfectly fit our computations, in the sense that $S(w_1, w_2, w_3)$ always returned to us a positive integer. This is a again substantial confirmation for both our computer program and for the remarkable precision of Arthur's results. This also gives some mysterious significance for the first non-trivial invariants of the group $W(E_7)^+$. One deduces in particular Table 7, and from this table Corollary 1.5 of the introduction.

Corollary 5.7.** *If $w_1 < 23$ then $S(w_1, w_2, w_3) = 0$. There are exactly 7 triples $(23, w_2, w_3)$ such that $S(23, w_2, w_3) \neq 0$, and for each of them $S(23, w_2, w_3) = 1$.*

As far as we know, none of these 7 automorphic representations (symplectic of rank 6) had been discovered before. As explained in the introduction, they are related to the 121 Borcherds even lattices of rank 25 and covolume $\sqrt{2}$, in the same way as the

4 Tsushima's forms $\Delta_{19,7}$, $\Delta_{21,5}$, $\Delta_{21,9}$ and $\Delta_{21,13}$ are related to Niemeier lattices, as discovered in [ChLa]. It would be interesting to know more about those forms, e.g. some of their $c_p(\pi)$. Our tables actually reveals a number of triples (w_1, w_2, w_3) such that $S(w_1, w_2, w_3) = 1$. For all of them we have thus $P_p(\Delta_{w_1, w_2, w_3}) \in \mathbb{Z}[T]$ for each prime p (see the introduction for the definition of $P_p(\pi)$).

One obtains as well a complete endoscopic description of each $\Pi_{\underline{w}}(\mathrm{SO}(7))$. For instance Tables 12 and 13 describe entirely the set $\Pi_{w_1, w_2, w_3}(\mathrm{SO}(7))$ for $w_1 \leq 25$ whenever it is non-empty.

Let us explore some examples. It follows from case (iii) above that the number of $\pi \in \Pi_{\underline{w}}(\mathrm{SO}(7))$ such that $\psi(\pi)$ has the form $\pi_1[3]$ is $\delta_{w_1=w_3+4} \cdot S(w_2)$. For instance the first such π is $\Delta_{11}[3]$ which thus belongs to $\Pi_{13,11,9}(\mathrm{SO}(7))$. Our computations gives $m(13, 11, 9) = 1$ (hence nonzero!) which is not only in accord with Arthur's result but also says that

$$\Pi_{13,11,9}(G) = \{\Delta_{11}[3]\}.$$

The triple $\underline{w} = (13, 11, 9)$ turns out to be the first triple $\neq (5, 3, 1)$ such that $m(\underline{w}) \neq 0$. Our table even shows that

$$\forall 3 \leq u \leq 25, \quad m(u+2, u, u-2) = S(u),$$

which describes entirely $\Pi_{u+2, u, u-2}(\mathrm{SO}(7))$ for those u . One actually has

$$m(29, 27, 25) = 4 > S(27) = 2.$$

Let us determine $\Pi_{29,27,25}(\mathrm{SO}(7))$. We already found two forms $\Delta_{27}^2[3]$ (there are two elements in $\Pi_{\mathrm{alg}}(\mathrm{PGL}(2))$ of Hodge number 27). On the other hand, one checks from Tsushima's result that $S(29, 25) = 1$, so that there is a unique element in $\Delta_{29,25} \in \Pi_{\mathrm{alg}}^s(\mathrm{PGL}(4))$ with Hodge numbers $29 > 25$. The missing two elements are thus the two $\Delta_{29,25} \oplus \Delta_{27}^2$. Indeed, we are here in the endoscopic case (v) : 27 is between 25 and 29.

As another example, consider now the $\pi \in \Pi_{\underline{w}}(G)$ such that $\psi(\pi)$ has the form $\pi_1[2]$ (endoscopic case (ii)). There are exactly

$$\delta_{w_3=1} \cdot \delta_{w_1-w_2=2} \cdot \delta_{w_1 \equiv 1 \pmod{4}} \cdot S\left(\frac{w_1+1}{2}\right)$$

such π 's. The first one is thus $\mathrm{Sym}^2 \Delta_{11}[2]$ which belongs to $\Pi_{23,21,1}(\mathrm{SO}(7))$. Our computations gives $m(23, 21, 1) = 1$, which is not only in accord with Arthur's result but also says that

$$\Pi_{23,21,1}(\mathrm{SO}(7)) = \{\mathrm{Sym}^2 \Delta_{11}[2]\}.$$

6. DESCRIPTION OF $\Pi_{\text{disc}}(\text{SO}(9))$ AND $\Pi_{\text{alg}}^s(\text{PGL}(8))$

6.1. The semisimple \mathbb{Z} -group $\text{SO}(9)$. Consider the semisimple classical \mathbb{Z} -group

$$G = \text{SO}(9),$$

i.e. the special orthogonal group of the root lattice $L = A_1 \oplus E_8$ of the root system of that name, equipped with its canonical positive definite integral quadratic form. Let $W(E_8)$ denote the Weyl group of the root system of E_8 and let $\varepsilon : W(E_8) \rightarrow \{\pm 1\}$ be the signature. There is a natural homomorphism $W(E_8) \rightarrow G(\mathbb{Z})$, if we let $W(E_8)$ act on $A_1 \oplus E_8$ as $w \mapsto (w, \varepsilon(w))$ (§ 2.5 case III). One has $O(L) = \{\pm 1\} \times W(E_8)$, thus a natural isomorphism

$$W(E_8) \xrightarrow{\sim} G(\mathbb{Z}).$$

The group W has order

$$|W(E_8)| = 8! \cdot 2^5 \cdot 3^3 \cdot 5 = 696729600$$

and the natural map $W(E_8) \rightarrow \text{SO}_{E_8}(\mathbb{F}_2)$ is surjective with kernel $\{\pm 1\}$ ([Bki]).

The class set $\text{Cl}(G) \simeq X_9$ has one element as $X_9 = \{A_1 \oplus E_8\}$ (§ 3.1, § 3.4). By Arthur's multiplicity formula, each $\pi \in \Pi_{\text{disc}}(G)$ has multiplicity 1. It follows from Prop. 3.6 that the number $m(V)$ of $\pi \in \Pi_{\text{disc}}(G)$ such that π_∞ is a given irreducible representation of $G(\mathbb{R})$ is

$$m(V) = \dim V^{W(E_8)^+},$$

which is exactly the number computed in the first chapter § 2.5 Case III. We refer to Table 4 for a sample of results.

The dual group of $\text{SO}(9)$ is $\widehat{G} = \text{Sp}(8, \mathbb{C})$.

6.2. Endoscopic partition of Π_w . We proceed in a similar way as in § 5.5.

Fix $\underline{w} = (w_1, w_2, w_3, w_4)$ with $w_1 > w_2 > w_3 > w_4$ odd positive integers. Fix as well once and for all a global Arthur parameter

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{glob}}(G)$$

such that the semisimple conjugacy class $\text{St}(z_{\psi_\infty})$ in $\mathfrak{sl}(8, \mathbb{C})$ has the eigenvalues

$$\{\pm w_i, 1 \leq i \leq 4\}.$$

Let us denote by π the unique element in $\Pi(\psi)$. We shall make explicit Arthur's multiplicity formula for $m(\pi)$, which is either 0 or 1, as in § 3.23.1. Recall the groups

$$C_\psi \subset C_{\psi_\infty} \subset \widehat{G} = \text{Sp}(8, \mathbb{C}).$$

6.2.1. The stable cases. This is the case $k = 1$, i.e. $C_\psi = Z$, for which the multiplicity formula trivially gives $m(\pi) = 1$. One has $\psi(\pi) = \pi_1[d_1]$ with $d_1|8$, $\pi_1 \in \Pi_{\text{alg}}^\perp(\text{PGL}(8/d_1))$, and $(-1)^{d_1-1}s(\pi_1) = -1$.

Case (i): (*tempered case*) $\psi = \pi_1$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(8))$, this is the unknown we want to compute.

Case (ii) : $\psi = \pi_1[2]$ where $\pi_1 \in \Pi_{\text{alg}}^o(\text{PGL}(4))$, say of Hodge numbers $u > v$ (recall u, v even and $u + v \equiv 2 \pmod{4}$). This occurs if and only if w has the form $(u+1, u-1, v+1, v-1)$. Recall that $\pi_1 = \pi' \otimes \pi''$ for a unique pair $\pi', \pi'' \in \Pi_{\text{alg}}(\text{PGL}(2))$ with respective Hodge numbers $(u+v)/2$ and $(u-v)/2$.

Case (iii) : $\psi = [8]$. This occurs if and only if $\underline{w} = (7, 5, 3, 1)$, and π is then the trivial representation of G .

6.2.2. Endoscopic cases of type $(n_1, n_2) = (6, 2)$. In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $C_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z . One will have to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\text{Min}(d_1, d_2)}$ in each case. Recall that $\rho^\vee : C_{\psi_\infty} \rightarrow \{\pm 1\}$ is the fundamental character defined in § 3.23.1. There are 6 cases.

Case (iv) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(6))$ and $\pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$. Denote by $a > b > c$ the Hodge numbers of π_1 and by d the Hodge number of π_2 . One has $\{a, b, c, d\} = \{w_1, w_2, w_3, w_4\}$. Moreover $\varepsilon_\psi(s_1) = 1$ as all the d_i are 1 (tempered case), so $m(\pi) = 1$ iff $\rho^\vee(s_1) = 1$, i.e. if $d \in \{w_1, w_3\}$. In other words,

$$m(\pi) = 1 \iff d > a > b > c \text{ or } a > b > d > c.$$

Case (v) : $d_1 = 1, d_2 = 2$, i.e. $\psi = \pi_1 \oplus [2]$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(6))$ has Hodge numbers $w_1 > w_2 > w_3$, with $w_3 > 1$. One sees that $\rho^\vee(s_1) = -1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_1) = (-1)^{(w_1+w_2+w_3+3)/2}$, it follows that

$$m(\pi) = 1 \iff w_1 + w_2 + w_3 \equiv 3 \pmod{4}.$$

Case (vi) : $d_1 = 2, d_2 = 1$, i.e. $\psi = \pi_1[2] \oplus \pi_2$ where $\pi_1 \in \Pi_{\text{alg}}^o(\text{PGL}(3))$ and $\pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$. Denote by a and b the respective Hodge numbers of π_1 and π_2 , so that $\{w_1, w_2, w_3, w_4\} = \{a+1, a-1, b, 1\}$. There are two cases : either $b > a$ or $b < a$. One sees that $\rho^\vee(s_1) = 1$ in both cases. On the other hand

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = -(-1)^{\frac{b+1}{2} + \text{Max}(a, b)}.$$

It follows that

$$m(\pi) = 1 \iff \begin{cases} b \equiv 3 \pmod{4}, & \text{if } b > a + 1, \\ b \equiv 1 \pmod{4}, & \text{if } b < a - 1. \end{cases}$$

Case (vii): $d_1 = 3, d_2 = 1$, i.e. $\psi = \pi_1[3] \oplus \pi_2$ where $\pi_1, \pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$. Denote by a and b the respective Hodge numbers of π_1 and π_2 , so that $\{w_1, w_2, w_3, w_4\} =$

$\{a+1, a, a-1, b\}$. One sees that $\rho^\vee(s_1) = 1$ if $b > a+1$, -1 otherwise. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = 1$. It follows that

$$m(\pi) = 1 \iff b > a+1.$$

Case (viii): $d_1 = 3, d_2 = 2$, i.e. $\psi = \pi_1[3] \oplus [2]$ where $\pi_1 \in \Pi_{\text{alg}}(\text{PGL}(2))$. The Hodge number of π_1 is thus $w_2 > 3$. We have $\rho^\vee(s_1) = -1$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1)^2 = 1$. It follows that

$$m(\pi) = 0$$

Case (ix) : $d_1 = 6, d_2 = 1$, i.e. $\psi = [6] \oplus \pi_2$ where $\pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$ has Hodge numbers w_1 with $w_1 > 5$. One sees that $\rho^\vee(s_1) = 1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_2) = (-1)^{(w_1+1)/2}$, it follows that

$$m(\pi) = 1 \iff w_1 \equiv 3 \pmod{4}.$$

Remark 6.3. Observe that the case $d_1 = d_2 = 2$, i.e. $\psi = \pi_1[2] \oplus [2]$ where $\pi_1 \in \Pi_{\text{alg}}^o(\text{PGL}(3))$, is impossible as it implies $w_2 = w_1 = 1$.

6.3.1. *Endoscopic cases of type $(n_1, n_2, n_3) = (4, 2, 2)$.* In this case $k = 3$, and C_ψ is generated by Z (or s_3) and s_1, s_2 . There are three cases.

Case (x) : (*tempered case*) $d_i = 1$ for each i , i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(4))$ and $\pi_2, \pi_3 \in \Pi_{\text{alg}}(\text{PGL}(2))$. Denote by $a > b$ the Hodge numbers of π_1 and by c and d the ones of π_2, π_3 , assuming $c > d$. Of course ε_ψ is trivial here, so $m(\pi) = 1$ if and only if ρ^\vee is trivial on $C_\psi = C_{\psi_\infty}$. One thus obtains

$$m(\pi) = 1 \iff c > a > d > b.$$

Case (xi): $d_1 = d_2 = 1$ and $d_3 = 2$, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus [2]$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(4))$, $\pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$ have respective Hodge numbers $a > b$ and c , with $\{w_1, w_2, w_3, w_4\} = \{a, b, c, 1\}$. If $a > c > b$ then $\rho^\vee(s_1) = 1$ and $\rho^\vee(s_2) = -1$, otherwise $\rho^\vee(s_1) = -1$ and $\rho^\vee(s_2) = 1$. On the other hand for $i = 1, 2$ one has $\varepsilon_\psi(s_i) = \varepsilon(\pi_i)$. It follows that

$$m(\pi) = 1 \iff \begin{cases} (a+b, c) \equiv (2, 1) \pmod{4}, & \text{if } a > c > b \\ (a+b, c) \equiv (0, 3) \pmod{4}, & \text{otherwise.} \end{cases}$$

Case (xii): $d_1 = 4, d_2 = d_3 = 1$, i.e. $\psi = [4] \oplus \pi_2 \oplus \pi_3$ where $\pi_2, \pi_3 \in \Pi_{\text{alg}}(\text{PGL}(2))$ with respective Hodge numbers w_1 and w_2 , with $w_2 > 3$. One has $\rho^\vee(s_2) = 1, \rho^\vee(s_3) = -1, \varepsilon_\psi(s_i) = \varepsilon(\pi_i)$ for $i = 2, 3$, thus

$$m(\pi) = 1 \iff (w_1, w_2) \equiv (3, 1) \pmod{4}.$$

6.3.2. *Endoscopic cases of type $(n_1, n_2, n_3, n_4) = (2, 2, 2, 2)$.* In this case $k = 4$, and C_ψ is generated by Z (or s_4) and s_1, s_2, s_3 . There are two cases.

Case (xiii) : (*tempered case*) $d_i = 1$ for each i , i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4$ where $\pi_i \in \Pi_{\text{alg}}(\text{PGL}(2))$ has Hodge number w_i . As ε_ψ is trivial but not ρ^\vee on $C_\psi = C_{\psi_\infty}$ we have in all cases

$$m(\pi) = 0.$$

Case (xiv) : $d_4 = 2$ and $d_1 = d_2 = d_3 = 1$, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus [2]$ where $\pi_i \in \Pi_{\text{alg}}(\text{PGL}(2))$ has Hodge number w_i , and $w_3 > 1$. One has $\rho^\vee(s_1) = \rho^\vee(s_3) = 1$ and $\rho^\vee(s_2) = -1$. On the other hand $\varepsilon_\psi(s_i) = \varepsilon(\pi_i)$ for $i = 1, 2, 3$. It follows that

$$m(\pi) = 1 \iff (w_1, w_2, w_3) \equiv (3, 1, 3) \pmod{4}.$$

6.3.3. *Endoscopic cases of type $(n_1, n_2) = (4, 4)$.* In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $C_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z . One only has to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\text{Min}(d_1, d_2)}$ in each case. .

Case (xv): (*tempered case*) $d_1 = d_2 = 1$, i.e. $\psi = \pi_1 \oplus \pi_2$ with $\pi_1, \pi_2 \in \Pi_{\text{alg}}^s(\text{PGL}(4))$. Let $a > b$ be the Hodge number of π_1 and $c > d$ the ones of π_2 , one may assume that $a > c$, i.e. $a = w_1$. As $\varepsilon_\psi = 1$, one sees that

$$m(\pi) = 1 \iff a > c > b > d.$$

Case (xvi): $d_1 = 1$ and $d_2 = 4$, i.e. $\psi = \pi_1 \oplus [4]$ where $\pi_1 \in \Pi_{\text{alg}}^s(\text{PGL}(4))$ has Hodge numbers $w_1 > w_2$ with $w_2 > 3$. It follows that $\rho^\vee(s_1) = -1$, and as $\varepsilon_\psi(s_1) = \varepsilon(\pi_1)$ one obtains

$$m(\pi) = 1 \iff w_1 + w_2 \equiv 0 \pmod{4}.$$

6.4. Conclusions. The inspection of each case above, and our previous computation of $S(w)$, $S(w, v)$, $S(w, v, u)$, $O^*(w)$ and $O(w, v)$, allow to compute the contribution of each endoscopic type except one, namely the stable and tempered type, which is actually $S(w_1, w_2, w_3, w_4)$, that we thus deduce from our computation of $m(w_1, w_2, w_3, w_4)$. The Corollary 1.6 and Table 8 follow from these computations.

Corollary 6.5.** *If $w_1 < 25$ then $S(w_1, w_2, w_3, w_4) = 0$. There are 33 triples (w_2, w_2, w_4) such that $S(25, w_2, w_3, w_4) \neq 0$, and in each case $S(25, w_2, w_3, w_4) = 1$.*

We refer to Table 14 for the description of all the nonempty $\Pi_{\underline{w}}(\text{SO}(9))$ when $w_1 \leq 23$.

For the application to Theorem 1.14, consider for instance the problem of describing $\Pi_{27,23,9,1}(\text{SO}(9))$. Our program tells us that

$$m(27, 23, 9, 1) = 5,$$

so that $|\Pi_{27,23,9,1}(\text{SO}(9))| = 5$. Fix $\pi \in \Pi_{27,23,9,1}(\text{SO}(9))$ and let $\psi(\pi) = (k, (n_i), (d_i), (\pi_i))$.

Assume first that $\psi(\pi)$ is not tempered, i.e. that some $d_i \neq 1$. We may assume that $d_k > 1$. One sees that $k > 1$, $d_k = 2$ and $d_i = 1$ for $i < k$. As $S(9) = 0$ we have $k \leq 3$. If $k = 2$ then we are in case (v). As $27 + 23 + 9 \equiv 3 \pmod{4}$ one really has to compute

$S(27, 23, 9)$. Our computer program tells us that $m(27, 23, 9) = 4$. On the other hand one has

$$S(27, 23, 9) = m(27, 23, 9) - S(27, 9) \cdot S(23)$$

by § 5.6. By Tsushima's formula we have $S(27, 9) = 1$. As $S(23) = 2$ we obtain

$$S(27, 23, 9) = 2.$$

There are thus two representations $\Delta_{27,23,9}^2 \oplus [2]$ in $\Pi_{27,23,9,1}(\mathrm{SO}(9))$.

Assume now that $k = 3$, so we are in case (xi) and the Hodge numbers of π_1 are a and 9. As $a + 9 \equiv 0 \pmod{4}$, the multiplicity formula forces thus $a = 23$. Tsushima's formula shows that $S(23, 9) = 1$. As $S(27) = 2$ there are indeed two parameters $\Delta_{27}^2 \oplus \Delta_{23,9} \oplus [2]$ in case (xi), whose associated π each have multiplicity 1 by the multiplicity formula.

Suppose now that π is tempered, i.e. $d_i = 1$ for all i . The multiplicity formula shows that 1 and 23 are Hodge numbers of a same π_i , say π_{i_0} . But we already checked that $S(23, 1) = 0$ and $S(23, 9, 1) = 0$, and $S(9) = 0$, it follows that $k = 1$, i.e. π is stable.

Corollary 6.6.** $\Pi_{27,23,9,1}(\mathrm{SO}(9)) = \{\Delta_{27,23,9}^2 \oplus [2], \Delta_{27}^2 \oplus \Delta_{23,9} \oplus [2], \Delta_{27,23,9,1}\}.$

7. DESCRIPTION OF $\Pi_{\text{disc}}(\text{SO}(8))$ AND $\Pi_{\text{alg}}^o(\text{PGL}(8))$

7.1. The semisimple \mathbb{Z} -group $\text{SO}(8)$. Consider the semisimple classical \mathbb{Z} -group

$$G = \text{SO}(8) = \text{SO}_{E_8},$$

i.e. the special orthogonal group of the root lattice E_8 equipped with its canonical positive definite integral quadratic form. Recall that $W(E_8)$ denote the Weyl group of the root system of E_8 , that $\varepsilon : W(E_8) \rightarrow \{\pm 1\}$ is the signature and that $W(E_8)^+ = \text{Ker } \varepsilon$. As the Dynkin diagram of E_8 has no non-trivial automorphism one has $O(E_8) = W(E_8)$, thus

$$G(\mathbb{Z}) = W(E_8)^+.$$

The class set $\text{Cl}(G) \simeq \tilde{X}_8$ has one element as $X_8 = \{E_8\}$ and of course $O(E_8) \neq \text{SO}(E_8)$ (§ 3.1, § 3.4).

We shall consider quadruples $\underline{w} = (w_1, w_2, w_3, w_4)$ where $w_1 > w_2 > w_3 > w_4 \geq 0$ are even integers. It is not necessary to consider the (w_1, w_2, w_3, w_4) with $w_4 < 0$ as $O(E_8)(\mathbb{Z}) = W$ contains root reflexions. Fix such a reflexion s . Then s acts by conjugation on $\mathcal{L}(G)$, hence on $\Pi_{\text{disc}}(G)$, with the following property : if π_∞ has the highest weight (n_1, n_2, n_3, n_4) , then $s(\pi)_\infty$ has the highest weight $(n_1, n_2, n_3, -n_4)$. Moreover $m(s(\pi)) = m(\pi)$.

Consider $m'(\underline{w}) = \sum_{\pi \in \Pi_{\underline{w}}(G)} m(\pi)$. It follows from Prop. 3.6 that

$$m'(\underline{w}) = \dim U_{\underline{w}}^{W(E_8)^+},$$

which is exactly the number computed in the first chapter § 2.5 Case II. We refer to Table 3 for a sample of results.

By Arthur's multiplicity formula, for each $\pi \in \Pi_{\underline{w}}(G)$ we have $m(\pi) + m(s(\pi)) \leq 2$. In particular, if $\underline{w} = (w_1, w_2, w_3, w_4)$ is such that $w_4 \neq 0$, then $m(\pi) = 1$. In this case, it follows that

$$m(\underline{w}) = m'(\underline{w}) = \dim U_{\underline{w}}^{W(E_8)}.$$

(Recall that $m(\underline{w}) = |\Pi_{\underline{w}}(G)|$).

The dual group of $\text{SO}(8)$ is $\widehat{G} = \text{SO}(8, \mathbb{C})$.

7.2. Endoscopic partition of Π_w . We proceed again in a similar way as in § 5.5.

Fix $\underline{w} = (w_1, w_2, w_3, w_4)$ with $w_1 > w_2 > w_3 > w_4 \geq 0$ even integers. Fix as well once and for all a global Arthur parameter

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\text{glob}}(G)$$

such that the semisimple conjugacy class $\text{St}(z_{\psi_\infty})$ in $\mathfrak{sl}(8, \mathbb{C})$ has the eigenvalues

$$\{\pm w_i, 1 \leq i \leq 4\}.$$

We shall make explicit Arthur's multiplicity formula for the number

$$m'(\psi) = \sum_{\pi \in \Pi(\psi) \cap \Pi_{\underline{w}}(G)} m(\pi),$$

following §3.23.2. It will be convenient to introduce the number

$$e(\underline{w}) = \begin{cases} 1 & \text{if } w_4 > 0, \\ 2 & \text{otherwise.} \end{cases}$$

Recall also the groups

$$C_\psi \subset C_{\psi_\infty} \subset \widehat{G} = \mathrm{SO}(8, \mathbb{C}).$$

Denote by $J \subset \{1, \dots, k\}$ the set of integers j such that $n_j \equiv 1 \pmod{2}$. It follows from Lemma 3.16 that :

- (i) If $j \notin J$ then $n_j \equiv 0 \pmod{4}$.
- (ii) $|J| = 0$ or 2 , and in this latter case $\sum_{j \in J} n_j \equiv 0 \pmod{4}$.

We will say that ψ is *even-stable* if $k = 1$, and *odd-stable* if $k = 2$ and $J = \{1, 2\}$.

7.2.1. The even-stable cases. We have $C_\psi = Z$ so the multiplicity formula trivially gives $m'(\psi) = e(\underline{w})$. One has $\psi(\pi) = \pi_1[d_1]$ with $d_1 | 8$, $\pi_1 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}(8/d_1))$, and $(-1)^{d_1-1} s(\pi_1) = -1$.

Case (i): (*tempered case*) $\psi = \pi_1$ where $\pi_1 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(8))$, this is the first unknown we want to compute.

Case (ii) : $\psi = \pi_1[2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^s(\mathrm{PGL}(4))$. This occurs if and only if $w_1 - w_2 = w_3 - w_4 = 2$ and π_1 has Hodge numbers $w_1 - 1, w_3 - 1$.

Case (iii) : $\psi = \pi_1[4]$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}(2))$. This occurs if and only if $w_1 = w_4 + 6$ and π_1 has Hodge numbers $w_1 - 3$.

7.2.2. The odd-stable cases. We have again $C_\psi = Z$ so the multiplicity formula trivially gives $m'(\psi) = 1$. One has $\psi(\pi) = \pi_1[d_1] \oplus \pi_2[d_2]$ with n_1, n_2, d_1 and d_2 odd. These cases only occur when $w_4 = 0$.

Case (iv): $d_1 = n_2 = 1$, i.e. $\psi = \pi_1 \oplus [1]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(7))$, which is the second unknown we want to compute.

Case (v) : $d_1 = d_2 = 1, n_1 = 5$, i.e. $\psi = \pi_1 \oplus \pi_2$ where $\pi_1 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(5))$ and $\pi_2 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(3))$.

Case (vi) $d_1 = n_1 = 5, d_2 = 1$, i.e. $\psi = [5] \oplus \pi_2$ where $\pi_2 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(3))$. In this case $w_2 = 4$.

Case (vii) $d_1 = 1, n_1 = 5, d_2 = 3$, i.e. $\psi = \pi_1 \oplus [3]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^o(\mathrm{PGL}(5))$. In this case $w_3 = 2$.

Case (viii): $d_1 = 7, n_2 = 1$, i.e. $\psi = [7] \oplus [1]$. This occurs if and only if $\underline{w} = (6, 4, 2, 0)$ and π is then the trivial representation of G .

7.2.3. *Endoscopic cases of type $(n_1, n_2) = (4, 4)$.* In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $C_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 (or s_2) and the center Z . One will have to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\text{Min}(d_1, d_2)}$ in each case. Recall that $\rho^\vee : C_{\psi_\infty} \rightarrow \{\pm 1\}$ is the fundamental character defined in § 3.23.2.

Case (ix) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1, \pi_2 \in \Pi_{\text{alg}}^o(\text{PGL}(4))$. Denote by $a > b$ the Hodge numbers of π_1 and by $c > d$ the ones of π_2 . We may assume $a > c$. One has $\{a, b, c, d\} = \{w_1, w_2, w_3, w_4\}$. Moreover $\varepsilon_\psi(s_1) = 1$ as all the d_i are 1 (tempered case), so $m'(\psi) \neq 0$ iff $\rho^\vee(s_1) = 1$, i.e. if $a > c > b > d$. In other words,

$$m'(\psi) = \begin{cases} e(\underline{w}) & \text{if } a > c > b > d, \\ 0 & \text{otherwise.} \end{cases}$$

Case (x) : $d_1 = 2, d_2 = 1$, i.e. $\psi = \pi_1[2] \oplus \pi_2$ where $\pi_1 \in \Pi_{\text{alg}}(\text{PGL}(2))$ and $\pi_2 \in \Pi_{\text{alg}}^o(\text{PGL}(4))$. If a is the Hodge number of π_1 and $b > c$ are the Hodge numbers of π_2 then $\{w_1, w_2, w_3, w_4\} = \{a+1, a-1, b, c\}$. One has

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = (-1)^{\text{Max}(a, b) + \text{Max}(a, c)}.$$

On the other hand $\rho^\vee(s_1) = -1$. It follows that

$$m'(\psi) = \begin{cases} e(\underline{w}) & \text{if } b > a > c, \\ 0 & \text{otherwise.} \end{cases}$$

Case (xi) : $d_1 = d_2 = 2$, i.e. $\psi = \pi_1[2] \oplus \pi_2[2]$ where $\pi_1, \pi_2 \in \Pi_{\text{alg}}(\text{PGL}(2))$ have respective Hodge numbers $w_1 - 1$ and $w_3 - 1$. One has $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = 1$ and $\rho^\vee(s_1) = -1$. It follows that

$$m'(\psi) = 0$$

in all cases.

7.2.4. *Endoscopic cases of type $(n_1, n_2, n_3) = (4, 3, 1)$.* In this case $k = 3, w_4 = 0$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2] \oplus [1]$$

and $C_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z . We have

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\text{Min}(d_1, d_2)} \varepsilon(\pi_1).$$

Case (xii) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1 \in \Pi_{\text{alg}}^o(\text{PGL}(4))$ and $\pi_2 \in \Pi_{\text{alg}}^o(\text{PGL}(3))$. Denote by $a > b$ the Hodge numbers of π_1 and c the one of π_2 . One has $\{a, b, c\} = \{w_1, w_2, w_3\}$ and $\varepsilon_\psi = 1$. The multiplicity is thus nonzero if and only if $\rho^\vee(s_1) = 1$, i.e. $a > c > b$:

$$m'(\psi) = \begin{cases} 1 & \text{if } a > c > b, \\ 0 & \text{otherwise.} \end{cases}$$

Case (xiii) : $d_1 = 2, d_2 = 1$, i.e. $\pi_1 \in \Pi_{\text{alg}}(\text{PGL}(2))$ and $\pi_2 \in \Pi_{\text{cusp}}^o(\text{PGL}(3))$. If a is the Hodge number of π_1 and if b is the one of π_2 , then $\{a+1, a-1, b\} = \{w_1, w_2, w_3\}$.

One has $\rho^\vee(s_1) = -1$. On the other hand, $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)\varepsilon(\pi_1) = (-1)^{\text{Max}(a,b)+1}$. It follows that

$$m'(\psi) = \begin{cases} 1 & \text{if } b > a, \\ 0 & \text{otherwise.} \end{cases}$$

Case (xiv) : $d_1 = 1, d_2 = 3$, i.e. $\psi = \pi_1 \oplus [3] \oplus [1]$ with $\pi \in \Pi_{\text{alg}}^o(\text{PGL}(4))$ of Hodge numbers $w_1 > w_2$ (here $w_3 = 2$). We have $\varepsilon_\psi = 1$ and $\rho^\vee(s_1) = -1$, so

$$m'(\psi) = 0$$

in all cases.

Case (xv): $d_1 = 2, d_2 = 3$, i.e. $\psi = \pi_1[2] \oplus [3] \oplus [1]$ with $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$ of Hodge number $a = w_1 - 1$. We have $\varepsilon_\psi(s_1) = \varepsilon(\pi_1) = (-1)^{\frac{a+1}{2}}$ and $\rho^\vee(s_1) = -1$, so

$$m'(\psi) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

7.3. Conclusions. The inspection of each case above, and our previous computation of $S(w)$, $S(w, v)$, $S(w, v, u)$, $O^*(w)$, $O(w, v)$ and $O^*(w, v)$, allow to compute the contribution of each endoscopic type except two, namely the even and odd stable and tempered types. The contribution of the even-stable tempered type is exactly

$$O(w_1, w_2, w_3, w_4)$$

when $w_4 \neq 0$, and $2 \cdot O(w_1, w_2, w_3, w_4)$ when $w_4 = 0$. The contribution of the odd-stable tempered type is

$$O^*(w_1, w_2, w_3).$$

This concludes the proof of Theorem 1.4. The Corollary 1.11 and Tables 9 and 10 follow from these computations.

Let us mention that we also have in our database the computation of the number of discrete automorphic representations of the non-connected group $O(8)$ of any given infinitesimal character. We shall not say more about this in this paper however.

8. DESCRIPTION OF $\Pi_{\text{disc}}(G_2)$

8.1. The semisimple definite G_2 over \mathbb{Z} . Consider the unique semisimple \mathbb{Z} -group G of type G_2 such that $G(\mathbb{R})$ is compact, namely the automorphism group scheme over \mathbb{Z} of "the" ring of Coxeter octonions (see [Gr1, §4]). We shall simply write G_2 for this \mathbb{Z} -group G . The reduction map $G_2(\mathbb{Z}) \rightarrow G_2(\mathbb{F}_2)$ is an isomorphism and

$$|G_2(\mathbb{Z})| = 2^6 \cdot 3^3 \cdot 7 = 12096.$$

The \mathbb{Z} -group G_2 admits a natural homomorphism into the \mathbb{Z} -group $\text{SO}(7)$ by its action on the lattice L of pure Coxeter octonions. For a well-chosen basis of $L[1/2]$, it follows from [CNP, §4] that the group $G_2(\mathbb{Z})$ becomes the subgroup of $\text{GL}(7, \mathbb{Z}[1/2])$ generated by the two elements

$$\frac{1}{2} \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

This allows not only to enumerate (with the computer) all the elements of $G_2(\mathbb{Z}) \subset \text{GL}(7, \mathbb{Z}[1/2])$ but to compute as well their characteristic polynomials. The list of the twelve obtained characteristic polynomials, together with the number of elements with that characteristic polynomial, is given in Table 1 (we denote by Φ_d the d -th cyclotomic polynomial). One easily checks for instance with this table that if $\chi(g, t)$ denotes the characteristic polynomial of g then

$$\frac{1}{12096} \sum_{g \in G_2(\mathbb{Z})} \chi(g, t) = t^7 - t^4 + t^3 - 1,$$

which is compatible with well-known fact that $\dim(\Lambda^3 L \otimes \mathbb{C})^{G_2(\mathbb{C})} = 1$.

TABLE 1. Characteristic polynomials of the elements of $G_2(\mathbb{Z}) \subset \text{SO}(7, \mathbb{R})$.

Char. Poly.	#	Char. Poly.	#
$\Phi_1 \Phi_3^3$	56	$\Phi_1 \Phi_3 \Phi_6^2$	504
$\Phi_1 \Phi_2^2 \Phi_4^2$	378	$\Phi_1^3 \Phi_4^2$	378
$\Phi_1^3 \Phi_2^4$	315	Φ_1^7	1
$\Phi_1 \Phi_2^2 \Phi_3 \Phi_6$	2016	$\Phi_1 \Phi_3 \Phi_{12}$	3024
$\Phi_1 \Phi_2^2 \Phi_8$	1512	$\Phi_1 \Phi_4 \Phi_8$	1512
$\Phi_1 \Phi_7$	1728	$\Phi_1^3 \Phi_3^2$	672

8.2. Polynomial invariants for $G_2(\mathbb{Z}) \subset G_2(\mathbb{R})$. To describe the finite dimensional representations of $G_2(\mathbb{R})$ we fix a maximal torus T and a system of positive roots Φ^+ of $(G_2(\mathbb{R}), T)$. Let $X = X^*(T)$, $X^\vee = X_*(T)$ and denote by $\langle \cdot, \cdot \rangle$ the canonical perfect pairing between them.

Let $\alpha, \beta \in X$ the simple roots in Φ^+ where α is short and β is long. The positive roots are thus

$$\alpha, \beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha, 2\beta + 3\alpha,$$

where $\alpha, \beta + \alpha$ and $\beta + 2\alpha$ are short, and $X = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. The inverse root system is again of type G_2 , with simple positive roots $\alpha^\vee, \beta^\vee \in X^\vee$ with α^\vee long, and where

$$\langle \alpha, \beta^\vee \rangle = -1 \text{ and } \langle \beta, \alpha^\vee \rangle = -3.$$

It follows that the dominant weights are the $a\alpha + b\beta$ where $a, b \in \mathbb{Z}$ satisfy $2b \geq a \geq 3b/2$. The fundamental representations with respective fundamental weights

$$\omega_1 = 2\alpha + \beta, \quad \omega_2 = 3\alpha + 2\beta$$

will be denoted by V_7 and V_{14} , because of their respective dimension 7 and 14. One easily checks that $V_7 = L \otimes \mathbb{C}$ and V_{14} is the adjoint representation. The half-sum of positive roots is $\rho = 5\alpha + 3\beta = \omega_1 + \omega_2$.

Definition 8.3. If $w > v$ are even non-negative integers, we denote by $U_{w,v}$ the irreducible representation of $G_2(\mathbb{R})$ with highest weight

$$\frac{w - v - 2}{2} \omega_1 + \frac{v - 2}{2} \omega_2.$$

We also denote by $\Pi_{w,v}(G_2)$ the subset of $\pi \in \Pi_{\text{disc}}(G_2)$ such that $\pi_\infty \simeq U_{w,v}$, and set $m(w, v) = \sum_{\pi \in \Pi_{w,v}} m(\pi)$.

This curious looking numbering has the following property. If

$$\varphi : W_{\mathbb{R}} \longrightarrow \widehat{G_2}$$

is the Langlands parameter of $U_{w,v}$, and if $\rho_7 : \widehat{G_2} \rightarrow \text{SO}(7, \mathbb{C})$ is the 7-dimensional irreducible representation of $\widehat{G_2}$, then $\rho_7 \circ \varphi$ is the representation

$$\mathbf{I}_{w+v} \oplus \mathbf{I}_w \oplus \mathbf{I}_v \oplus \varepsilon.$$

Indeed, the weights of ρ_7 are $0, \pm\beta^\vee, \pm(\alpha^\vee + \beta^\vee), \pm(\alpha^\vee + 2\beta^\vee)$.

Observe that $\rho_7 \circ \varphi$ determines the equivalence class of φ . This is a special case of the fact that the conjugacy class of any element $g \in G_2(\mathbb{R})$ (resp. of any semisimple element in $G_2(\mathbb{C})$) is uniquely determined by its characteristic polynomial in V_7 . Indeed, this follows from the identity

$$V_{14} \oplus V_7 \simeq \Lambda^2 V_7.$$

This property makes the embedding $G_2(\mathbb{C}) \subset \text{SO}(7, \mathbb{C})$ quite suitable to study G_2 and its subgroups. In particular, Table 1 leads to a complete determination of the semisimple conjugacy classes in $G_2(\mathbb{R})$ of the elements of $G_2(\mathbb{Z})$, which is the ingredient we need to apply the method of § 2.5.

Gross showed that $|\mathrm{Cl}(G_2)| = 1$ in [Gr1, §5], it follows that

$$m(w, v) = \dim U_{w,v}^{G_2(\mathbb{Z})}.$$

See Table 5 for a sample of computations. As we shall see below, one should have $m(\pi) = 1$ for each $\pi \in \Pi_{\mathrm{disc}}(G)$, and we thus expect that $m(w, v) = |\Pi_{w,v}(G_2)|$.

Automorphic forms for the \mathbb{Q} -group G_2 have been previously studied by Gross, Lansky, Pollack and Savin : see [GrS], [GrP], [LP] and [P]. Although most of the automorphic forms studied by those authors are Steinberg at one finite place, they may be trivial at the infinite place. Pollack and Lansky are also able to compute some Hecke eigenvalues in some cases.

8.4. Endoscopic classification of $\Pi_{\mathrm{disc}}(G_2)$. We recall Arthur's conjectural description of $\Pi_{\mathrm{disc}}(G_2)$, following his general conjecture in [A1]. Most of the results here will thus be conditional to the existence of the group $\mathcal{L}_{\mathbb{Z}}$ of §4.19 and to these conjectures, that we will make explicit. All the facts stated below about the structure of G_2 can be simply checked on its the root system. We refer to [GaGu1] for a complete analysis of Arthur's conjectures for the split groups of type G_2 , in a much greater generality than we actually need here, and for a survey of the known results.

A global discrete Arthur parameter for the \mathbb{Z} -group G_2 is a \widehat{G}_2 -conjugacy class of morphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow \widehat{G}_2$$

such that :

- (a) ψ has finite centralizer in \widehat{G}_2 ,
- (b) $\psi_{\infty} = \psi|_{\mathrm{W}_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C})}$ is an Adams-Johnson parameter for $G_2(\mathbb{R})$.

Observe that by property (b) the centralizer $C_{\psi_{\infty}}$ of ψ_{∞} in \widehat{G}_2 is an elementary abelian 2-group, hence so is the centralizer $C_{\psi} \subset C_{\psi_{\infty}}$ of ψ . As $\mathcal{L}_{\mathbb{Z}}$ is connected, observe that the Zariski-closure of $\mathrm{Im}(\psi)$ is a connected complex reductive subgroup of \widehat{G}_2 .

This severely limits the possibilities for $\mathrm{Im}(\psi)$. Up to conjugacy there are exactly 3 connected complex reductive subgroups of \widehat{G}_2 whose centralizers are elementary abelian 2-groups :

- (i) the group \widehat{G}_2 itself, with trivial centralizer,
- (ii) a principal $\mathrm{PGL}(2, \mathbb{C})$ homomorphism, again with trivial centralizer,
- (iii) the centralizer $H_s \simeq \mathrm{SO}(4, \mathbb{C})$ of an element s of order 2, whose centralizer is the center $\langle s \rangle$ of H_s .

Recall that up to conjugacy there a unique element s of order 2 in \widehat{G}_2 . The isomorphism $H_s \simeq \mathrm{SO}(4, \mathbb{C})$ in (iii) is actually canonical up to inner automorphisms as H_s is its own normalizer in \widehat{G}_2 . Indeed, one has two distinguished injective homomorphisms $\mathrm{SL}(2, \mathbb{C}) \rightarrow H_s$, one of which being a short radicial $\mathrm{SL}(2, \mathbb{C})$ and the other one being a long radicial $\mathrm{SL}(2, \mathbb{C})$ (the long and short roots being orthogonal)

We shall need some facts about the restrictions of V_7 and V_{14} to these groups. We denote by ν_{long} and ν_{short} the two 2-dimensional irreducible representations of H_s which are respectively non-trivial on the long and short $\text{SL}(2, \mathbb{C})$ inside H_s .

Lemma 8.5. *Let $s \in \widehat{G_2}$ be an element of order 2.*

- (i) $(V_7)_{|H_s} = \nu_{\text{long}} \otimes \nu_{\text{short}} \oplus \text{Sym}^2 \nu_{\text{short}},$
- (ii) $(V_{14})_{|H_s} = \text{Sym}^2 \nu_{\text{long}} \oplus \text{Sym}^2 \nu_{\text{short}} \oplus \text{Sym}^3 \nu_{\text{short}} \otimes \nu_{\text{long}}.$

Moreover, the restriction of V_7 to a principal $\text{PGL}(2, \mathbb{C})$ is isomorphic to $\nu_7 = \text{Sym}^6(\mathbb{C}^2)$.

If ψ is a global Arthur parameter for G_2 , then $\rho_7 \circ \psi$ actually defines a global Arthur parameter for $\text{Sp}(6)$, that we shall denote ψ^{SO} . The previous lemma and discussion show that the equivalent class of ψ^{SO} determines the conjugacy class of ψ .

Fix a global Arthur parameter ψ as above. We denote by

$$\pi(\psi) \in \Pi(G_2)$$

the unique representation π such that $c(\pi)$ is associated to ψ by the standard Arthur recipe. Explicitly, for each prime p we have $c_p(\pi) = \psi(\text{Frob}_p \times e_p)$ (see § 3.11), and π_∞ is the unique representation of $G_2(\mathbb{R})$ whose infinitesimal character is the one of the Langlands parameter φ_{ψ_∞} (assumption (b) on ψ). Arthur's conjectures describe $\Pi_{\text{disc}}(G_2)$ as follows. First, any $\pi \in \Pi_{\text{disc}}(G_2)$ should be of the form $\pi(\psi)$ for a unique ψ satisfying (a) and (b). Second, they describe $m(\pi(\psi))$ for each π as follows.

Case (i) : stable tempered cases, $\psi^{\text{SO}} \in \Pi_{\text{alg}}^\circ(\text{PGL}(7))$. This is when $C_\psi = 1$ and $\psi(\text{SL}(2, \mathbb{C})) = 1$. In this case

$$m(\pi) = 1.$$

By Prop.4.20, a $\pi \in \Pi_{\text{alg}}^\circ(\text{PGL}(7))$ has the form ψ^{SO} for a stable tempered ψ if and only if $c(\pi) \in \rho_7(\mathcal{X}(\widehat{G_2}(\mathbb{C})))$. It is equivalent to ask that $c(\pi) \times 1$, viewed as an element in $\mathcal{X}(\text{SO}(8, \mathbb{C}))$ is invariant by a triality automorphism. Moreover, $\text{Im}(\psi)$ is either G_2 or a principal $\text{PGL}(2)$. The latter case occur if and only if $\pi(\psi)_\infty \simeq U_{w,v}$ where $v \equiv 2 \pmod{4}$ and $w = 2v$, in which case it occurs exactly $S(v/2)$ times.

Case (ii) : stable non-tempered case, $\psi^{\text{SO}} = [7]$. Then π is the trivial representation, the unique element in $\Pi_{4,2}(G_2)$.

There are three other cases for which $\text{Im}(\psi) = H_s$. In those cases we have

$$C_\psi = \langle s \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

Arthur's multiplicity formula requires two ingredients. The first one is the character

$$\varepsilon_\psi : C_\psi \rightarrow \mathbb{C}^*$$

given by Arthur's general recipe [A1]. This character is trivial if $\psi(\text{SL}(2, \mathbb{C})) = 1$. Otherwise there are two distinct cases :

- (i) $\nu_{\text{short}} \circ \psi|_{\text{SL}(2, \mathbb{C})} = \nu_2$ and $\nu_{\text{long}} \circ \psi|_{\mathcal{L}_{\mathbb{Z}}} = r(\pi)$ for some $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$ (§4.19). Then

$$\varepsilon_{\psi}(s) = \varepsilon(\pi) = (-1)^{(w+1)/2}$$

where w is the Hodge number of π .

- (ii) $\nu_{\text{long}} \circ \psi|_{\text{SL}(2, \mathbb{C})} = \nu_2$ and $\nu_{\text{short}} \circ \psi|_{\mathcal{L}_{\mathbb{Z}}} = r(\pi)$ for some $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$. Then $\varepsilon_{\psi}(s) = \varepsilon(\text{Sym}^3 \pi)$. If w is the Hodge number of π , observe that

$$\varepsilon(\text{Sym}^3 \pi) = (-1)^{(w+1)/2 + (3w+1)/2} = -1$$

for each w , thus ε_{ψ} is the non-trivial character in this case.

Observe that the a priori remaining case $\psi(\mathcal{L}_{\mathbb{Z}}) = 1$ does not occur as property (b) is not satisfied for such a ψ (the infinitesimal character $z_{\psi_{\infty}}$ is not regular).

The second ingredient is the restriction to C_{ψ} of the character $\rho^{\vee} : C_{\psi_{\infty}} \rightarrow \mathbb{C}^*$. The multiplicity formula will then take the form : $m(\pi) = 1$ if $\rho^{\vee}(s) = \varepsilon_{\psi}(s)$ and $m(\pi) = 0$ otherwise.

In order to compute $\rho^{\vee}(s)$ we fix \widehat{T} a maximal torus in \widehat{G}_2 such that $X^*(\widehat{T}) = X^{\vee}$. Observe that the centralizer T' of $\rho_7(\widehat{T})$ in $\text{SO}(7, \mathbb{C})$ is a maximal torus of the latter group. We consider the standard root system Φ' for $(\text{SO}(7, \mathbb{C}), T')$ recalled in § 2.5, in particular $X^*(T') = \bigoplus_{i=1}^3 \mathbb{Z}e_i$. Then $\Phi^{\vee} = \Phi'_{|\widehat{T}}$ is a root system for $(\widehat{T}, \widehat{G}_2)$ with positive roots $(\Phi^{\vee})^+ = (\Phi')^+_{|\widehat{T}}$: up to conjugating ρ_7 we may thus assume that $(\Phi^{\vee})^+$ is the positive root system of § 8.2.

Lemma 8.6. *Under the assumptions above, we have $\rho^{\vee}(s) = e_2(\rho_7(s))$.*

Proof — Under the assumptions above, if $\lambda \in X_*(\widehat{T})$ is such that $\rho_7(\lambda)$ is $(\Phi')^+$ -dominant then λ is $(\Phi^{\vee})^+$ -dominant. We have already seen that the respective restriction to \widehat{T} of e_1, e_2, e_3 are the elements $2\beta^{\vee} + \alpha^{\vee}$, $\beta^{\vee} + \alpha^{\vee}$ and β^{\vee} . The lemma follows from the identity

$$\rho^{\vee} = 5\beta^{\vee} + 3\alpha^{\vee} \equiv (e_2)_{|\widehat{T}} \pmod{2X^*(\widehat{T})}.$$

□

We can now make explicit the three remaining multiplicity formulae.

Case (iii) : Tempered endoscopic case, i.e. $\psi^{\text{SO}} = \pi_{\text{long}} \otimes \pi_{\text{short}} \oplus \text{Sym}^2 \pi_{\text{short}}$ where $\pi_{\text{short}}, \pi_{\text{long}} \in \Pi_{\text{alg}}(\text{PGL}(2))$ have respective Hodge numbers $w_{\text{short}}, w_{\text{long}}$. Of course, $\text{Sym}^2 \pi_{\text{short}} \in \Pi_{\text{alg}}^o(\text{PGL}(3))$ has Hodge number $2w_{\text{short}}$ and $\pi_{\text{long}} \otimes \pi_{\text{short}} \in \Pi_{\text{alg}}^o(\text{PGL}(4))$ has Hodge numbers $w_{\text{short}} + w_{\text{long}}$ and $|w_{\text{short}} - w_{\text{long}}|$. We also have $\varepsilon_{\psi}(s) = 1$. But by Lemma 8.5 (i) we have $e_2(\rho_7(s)) = 1$ if and only if

$$w_{\text{long}} + w_{\text{short}} > 2w_{\text{short}} > w_{\text{long}} - w_{\text{short}},$$

thus $m(\pi) = 1$ in this case and $m(\pi) = 0$ otherwise.

Case (iv) : Non-tempered endoscopic case 1, $\psi^{\text{SO}} = \pi[2] \oplus \text{Sym}^2 \pi$ where $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$, say with Hodge number w . We have seen that in this case $\varepsilon_{\psi}(s) = -1$. On

the other hand $e_2(\rho_7(s)) = -1$ if and only if $w - 1 < 2w$, which is always satisfied as $w > 1$. Arthur's multiplicity formula tells us that

$$m(\pi) = 1$$

in all cases.

Case (v) : Non-tempered endoscopic case 2, $\psi^{\text{SO}} = \pi[2] \oplus [3]$ where $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$, say with Hodge number w . We have seen that $\varepsilon_\psi(s) = \varepsilon(\pi) = (-1)^{(w+1)/2}$. Observe that $e_2(\rho_7(s)) = -1$ as $w - 1 > 3$. Arthur's multiplicity formula tells us then that

$$m(\pi) = 1 \Leftrightarrow w \equiv 1 \pmod{4}.$$

Let us mention that the multiplicity formula for the Arthur's packets appearing in case (v) has been established for the split groups of type G_2 in [GaGu2].

8.7. Conclusions. The inspection of each case above and the well-known formula for $S(w)$ allow to compute the conjectural number $G_2(w, v)$ of $\pi \in \Pi_{\text{alg}}^o(\text{PGL}(7))$ such that $c(\pi) \in \rho_7(\mathcal{X}(\widehat{G_2}))$ and with Hodge numbers $w + v > w > v$. Concretely,

$$\begin{aligned} G_2(w, v) &= m(w, v) - \delta_{w=4} - O^*(w) \cdot O(w + v, v) \\ &\quad - \delta_{w-v=2} \cdot S(w - 1) - \delta_{v=2} \cdot \delta_{w \equiv 0 \pmod{4}} \cdot S(w + 1). \end{aligned}$$

See Table 11 for a sample of results when $w + v \leq 58$. The Sato-Tate group of each of the associated π is conjecturally the compact group of type $G_2 \subset \text{SO}(7, \mathbb{R})$ (rather than $\text{SO}(3, \mathbb{R})$ principally embedded in the latter) : this follows from Prop. 4.20 as the motivic weight of $\text{Sym}^6 \pi$ for $\pi \in \Pi_{\text{alg}}(\text{PGL}(2))$ is at least $66 > 58$.

9. APPLICATION TO SIEGEL MODULAR FORMS

9.1. Vector valued Siegel modular forms of level 1. We consider in this chapter the classical Chevalley \mathbb{Z} -group $\mathrm{Sp}(2g)$, whose dual group is $\mathrm{SO}(2g+1, \mathbb{C})$. Let

$$\underline{w} = (w_1, w_2, \dots, w_g)$$

where the w_i are even non-negative integers such that $w_1 > w_2 > \dots > w_g$. To such a \underline{w} we may associate a semisimple conjugacy class $z_{\underline{w}}$ in $\mathfrak{so}(2g+1, \mathbb{C})$, namely the class with eigenvalues $\pm \frac{w_i}{2}$ for $i = 1, \dots, g$, and 0. Recall that for any such \underline{w} , there is an L -packet of discrete series with infinitesimal characters $z_{\underline{w}}$, and that this L -packet contains two "holomorphic" discrete series which are outer conjugate by $\mathrm{PGSp}(2g, \mathbb{R})$. We make once and for all a choice for the holomorphic ones (hence for the anti-holomorphic as well).

Recall the space

$$S_{\underline{w}}(\mathrm{Sp}(2g, \mathbb{Z}))$$

of holomorphic vector valued Siegel modular forms with infinitesimal character $z_{\underline{w}}$. If (ρ, V) is the irreducible representation of $\mathrm{GL}_g(\mathbb{C})$ with standard highest weight $m_1 \geq m_2 \geq \dots \geq m_g$, and if $m_g > g$, recall that a (ρ, V) -valued Siegel modular form has infinitesimal character $z_{\underline{w}}$ where $\underline{w} = (w_i)$ and $w_i = 2(m_i - i)$ for each $i = 1, \dots, g$ (see e.g. [AS, §4.5]). Denote also

$$\Pi_{\underline{w}}(\mathrm{Sp}(2g))$$

the set of $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}(2g))$ such that π_{∞} is the holomorphic discrete series with infinitesimal character $z_{\underline{w}}$. Such a π is tempered at the infinite place, thus it follows from a result of Wallach [W, Thm. 4.3] that $\Pi_{\underline{w}}(\mathrm{Sp}(2g)) \subset \Pi_{\mathrm{cusp}}(\mathrm{Sp}(2g))$. By Arthur's multiplicity formula, $m(\pi) = 1$ for each $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}(2g))$, it is well-known that this implies

$$\dim S_{\underline{w}}(\mathrm{Sp}(2g, \mathbb{Z})) = |\Pi_{\underline{w}}(\mathrm{Sp}(2g))|.$$

By Lemma 3.16, Arthur's multiplicity formula allows to express $|\Pi_{\underline{w}}(\mathrm{Sp}(2g))|$ in terms of various $S(-)$, $O(-)$ and $O^*(-)$. We shall give now the two ingredients needed to make this computation in general and we shall apply them later in the special case $g = 3$.

9.2. Two lemmas on holomorphic discrete series. Let

$$\varphi_{\underline{w}} : W_{\mathbb{R}} \rightarrow \mathrm{SO}(2g+1, \mathbb{C})$$

be the discrete series Langlands parameter with infinitesimal character $z_{\underline{w}}$, and let

$$\Pi(\varphi_{\underline{w}})$$

be the associated L -packet of discrete series representations of $\mathrm{Sp}(2g, \mathbb{R})$ with infinitesimal character $z_{\underline{w}}$. Recall that the centralizer of $\varphi_{\underline{w}}(W_{\mathbb{C}})$ in $\mathrm{SO}(2g+1, \mathbb{C})$ is a maximal torus \widehat{T} in $\mathrm{SO}(2g+1, \mathbb{C})$ and that the centralizer $C_{\varphi_{\underline{w}}}$ of $\varphi_{\underline{w}}(W_{\mathbb{R}})$ is the 2-torsion subgroup of \widehat{T} . There is also a unique Borel subgroup $\widehat{B} \supset \widehat{T}$ for which the element $\lambda \in X_*(\widehat{T})[1/2]$ such that $\varphi_{\underline{w}}(z) = (z/\bar{z})^{\lambda}$ for all $z \in W_{\mathbb{C}}$ is dominant with respect to \widehat{B} .

We consider the setting and notations of §10.1 with $G = \mathrm{Sp}(2g, \mathbb{C})$. The strong forms $t \in \mathcal{X}_1(T)$ such that $G_t \simeq \mathrm{Sp}(2g, \mathbb{R})$ are the ones such that

$$t^2 = -1,$$

and they form a single W -orbit. Let us fix an isomorphism between $\mathrm{Sp}(2g, \mathbb{R})$ and $G_{[t]}$ for t in this W -orbit, which thus identify $\Pi(\varphi_{\underline{w}})$ with $\Pi(\varphi_{\underline{w}}, G_{[t]})$ (§10.5). This being done, Shelstad's parameterization gives a canonical injective map (see §10.7)

$$\tau : \Pi(\varphi_{\underline{w}}) \rightarrow \mathrm{Hom}(C_{\varphi_{\underline{w}}}, \mathbb{C}^*).$$

(We may replace $S_{\varphi_{\underline{w}}}$ by $C_{\varphi_{\underline{w}}}$ in the range as $\mathrm{Sp}(2g, \mathbb{R})$ is split, see Cor.10.14). Our first aim is to determine the image of π_{hol} and π_{ahol} , namely the holomorphic and anti-holomorphic discrete series in $\Pi(\varphi_{\underline{w}})$.

For a well-chosen \mathbb{Z} -basis (e_i) of $X^*(\widehat{T})$, the positive roots of $(\mathrm{SO}(2g+1, \mathbb{C}), \widehat{B}, \widehat{T})$ are $\{e_i, i = 1, \dots, g\} \cup \{e_i \pm e_j, 1 \leq i < j \leq g\}$ as in § 2.5. Let $(e_i^*) \in X^*(T)$ denote the dual basis of (e_i) . The set of positive roots of $(\mathrm{Sp}(2g, \mathbb{C}), B, T)$ dual to the positive root system above is the set $\{2e_i^*, i = 1, \dots, g\} \cup \{e_i^* \pm e_j^*, 1 \leq i < j \leq g\}$. If $t \in T$ we also write $t = (t_i)$ where $t_i = e_i^*(t)$ for each $i = 1, \dots, g$.

Lemma 9.3. *The Shelstad characters of π_{hol} and π_{ahol} are the restrictions to $C_{\varphi_{\underline{w}}}$ of the following elements of $X^*(\widehat{T})$:*

$$e_1 + e_3 + e_5 + \dots + e_{2[(g-1)/2]+1} \quad \text{and} \quad e_2 + e_4 + e_6 + \dots + e_{2[g/2]}.$$

Proof — Let $t \in \mathcal{X}_1(T)$ such that $t^2 = -1$. Recall that $\mathrm{Int}(t)$ is a Cartan involution of G_t and that K_t is the associated maximal compact subgroup of $G_t \simeq \mathrm{Sp}(2g, \mathbb{R})$. Let $\mathfrak{g} = \mathfrak{k}_t \oplus \mathfrak{p}$ the Cartan decomposition relative to $\mathrm{Int}(t)$. We have $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where $\mathfrak{p}_{\pm} \subset \mathfrak{p}$ are two distinct irreducible K_t -submodules for the adjoint action. As a general fact, the representation $\pi_t(\lambda)$ is a holomorphic or anti-holomorphic discrete series of G_t if and only if \mathfrak{b} is included in either $\mathfrak{k}_t \oplus \mathfrak{p}_+$ or in $\mathfrak{k}_t \oplus \mathfrak{p}_-$. In those cases, \mathfrak{k}_t is thus a standard Levi subalgebra of $(\mathfrak{g}, \mathfrak{b}, \mathfrak{t})$ isomorphic to \mathfrak{gl}_g . There is a unique such Lie algebra, namely the one with positive roots the $e_i^* - e_j^*$ for $i < j$. It follows that $\pi_t(\lambda)$ is a holomorphic discrete series if and only if the positive roots of T in \mathfrak{k}_t are the $e_i^* - e_j^*$ for each $1 \leq i < j \leq g$, i.e. if $t_i = t_j$ for $j \neq i$. As $t^2 = -1$, the two possibilities are thus the elements

$$t_+ = (i, i, \dots, i) \quad \text{and} \quad t_- = (-i, -i, \dots, -i).$$

We have $t_b = e^{i\pi\rho^\vee} = (i^{2g-1}, \dots, i, -i, i)$ (see §10.7), so $t_{\pm}t_b^{-1} = \pm(\dots, -1, 1, -1, 1)$. Let $\mu = e_1 + e_3 + \dots$ and $\mu' = e_2 + e_4 + \dots$ be the two elements of $X_*(T) = X^*(\widehat{T})$ given in the statement. One concludes as

$$e^{i\pi\mu} = (-1, 1, -1, \dots) \quad \text{and} \quad e^{i\pi\mu'} = (1, -1, 1, \dots).$$

□

The second ingredient is to determine which Adams-Johnson packets $\Pi(\psi)$ of $\mathrm{Sp}(2g, \mathbb{R})$ contains a holomorphic discrete series.

Lemma 9.4. *Let $\psi : W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SO}(2g+1, \mathbb{C})$ be an Adams-Johnson parameter for $\mathrm{Sp}(2g, \mathbb{C})$. Then $\widetilde{\Pi}(\psi)$ contains discrete series of $\mathrm{Sp}(2g, \mathbb{R})$ if and only if the underlying representation of $W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C})$ on \mathbb{C}^{2g+1} does not contain any $1 \otimes \nu_q$ or $\varepsilon \otimes \nu_q$ where $q > 1$.*

Furthermore, if ψ has this property then the holomorphic and anti-holomorphic discrete series of $\mathrm{Sp}(2g, \mathbb{R})$ belong to $\Pi(\psi)$.

Proof — Let T, B, L and λ be attached to ψ as in §10.2 and §10.5, recall that $L \subset \mathrm{Sp}(2g, \mathbb{C})$ is a Levi factor of a parabolic subgroup. From the last example of §10.2, from which we take the notations, we have

$$L \simeq \mathrm{Sp}(d-1, \mathbb{C}) \times \prod_{i \neq 0} \mathrm{GL}(d_i, \mathbb{C}).$$

Moreover, the underlying representation of $W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C})$ on \mathbb{C}^{2g+1} does not contain any $1 \otimes \nu_q$ or $\varepsilon \otimes \nu_q$ where $q > 1$ if and only if $d = 1$. On the other hand, $\Pi(\psi)$ contains a discrete series $\pi_t(\lambda)$ of $\mathrm{Sp}(2g, \mathbb{R})$ if and only if there is a $t \in \mathcal{X}_1(T) \cap Z(L)$ such that $t^2 = -1$, by Lemma 10.6. As $\mathrm{Sp}(d-1, \mathbb{C})$ does not contain any element of square -1 in its center for $d > 1$, the first assertion follows.

Assume now that L is a product of general complex linear groups. It is equivalent to ask that the positive roots of L with respect to (B, T) are among the $e_i^* - e_j^*$ for $i < j$. In particular, the element $t_0 = \pm(i, i, \dots, i) \in \mathcal{X}_1(T)$ is in the center of L , thus $\pi_{t_0, B}(\lambda) \in \tilde{\Pi}(\psi)$ by Lemma 10.6 and Lemma 10.9. But we have seen in the proof of Lemma 9.3 that this is a holomorphic/anti-holomorphic discrete series. \square

The difference between π_{hol} and π_{ahol} is not really meaningful for our purposes, and we will not need to say exactly which of the two characters in Lemma 9.3 corresponds to e.g. π_{hol} (of course this would be possible if we had defined π_{hol} more carefully). More importantly, let

$$\chi = \sum_{i=1}^g e_i$$

be the sum of the two elements of the statement of Lemma 9.3. Fix $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(\mathrm{Sp}(2g))$ with infinitesimal character $z_{\underline{w}}$, one has canonical embeddings

$$C_{\psi} \subset C_{\psi_{\infty}} \subset C_{\varphi_{\underline{w}}}$$

by §3.14 and §10.7, as $(\psi_{\infty})_{\mathrm{disc}} = \varphi_{\underline{w}}$. Recall that C_{ψ} is generated by elements s_i as in §3.20.

Lemma 9.5. *For each $i = 1, \dots, k$ such that n_i is even we have $\chi(s_i) = 1$.*

Proof — Indeed, it follows from Lemma 3.16 that if n_i is even then $n_i \equiv 0 \pmod{4}$. \square

Assume now that $\Pi_{\infty}(\psi)$ contains holomorphic discrete series, i.e. that $n_i \neq d_i$ for each i such that $n_i > 1$ by Lemma 9.4. In this case it follows from Lemma 10.9 that the characters of π_{hol} and π_{ahol} viewed as elements of $\Pi_{\infty}(\psi)$ are again the two characters of Lemma 9.3. But it follows then from the Lemma 9.5 above that the multiplicity formula is the same for the two $\pi \in \Pi(\psi)$ such that π_{∞} is either holomorphic or anti-holomorphic.

To conclude this paragraph let us say a word about the choice of the isomorphism that we fixed between $\mathrm{Sp}(2g, \mathbb{R})$ and $G_{[t]}$ (for $t^2 = -1$), which allowed to fix the parameterization τ . Consider for this the order 2 outer automorphism of $\mathrm{Sp}(2g, \mathbb{R})$ obtained as the conjugation by any element of $\mathrm{GSp}(2g, \mathbb{R})$ with similitude factor -1 . It defines in particular element defines an involution of $\Pi(\varphi_{\underline{w}}, G_{[t]})$ and we want to check the effect of this involution on Shelstad's parameterization. The next lemma shows that it is quite benign.

Lemma 9.6. *If $\pi \in \Pi(\varphi_{\underline{w}})$, then $\tau(\pi \circ \theta) = \tau(\pi) + \chi$.*

Proof — Fix some t such that $t^2 = -1$ and view θ as an outer automorphism of G_t . A suitable representant of θ in $\mathrm{Aut}(G_t)$ preserves (K_t, T_c) , and the automorphism of T_c obtained this way is well-defined up to $W(K_t, T_c)$. It is a simple exercise to check that it coincides here with the class of the inversion $t \mapsto t^{-1}$ of T_c . As $-1 \in W(G, T)$, it follows that $\pi_t(\lambda) \circ \theta = \pi_{t^{-1}}(\lambda)$. In other words, $\tau_0(\pi \circ \theta) = -\tau_0(\pi)$. As $\tau(\pi) = \tau_0(\pi) - \rho^\vee$, it follows that

$$\tau(\pi \circ \theta) = -\tau(\pi) - 2\rho^\vee.$$

But observe that $2\rho^\vee = \chi \bmod 2X^*(\hat{T})$. As $C_{\varphi_{\underline{w}}}$ is an elementary abelian two-group, the lemma follows. \square

It follows then from Lemma 9.5 that the choice of our isomorphism has no effect on the multiplicity formula for the $\pi \in \Pi(\psi)$ such that π_∞ is either holomorphic or anti-holomorphic.

9.7. An example : the case of genus 3. We shall now describe the endoscopic classification of $\Pi_{\underline{w}}(\mathrm{Sp}(6))$ for any $\underline{w} = (w_1, w_2, w_3)$. As an application, we will deduce in particular the following proposition stated in the introduction.

Proposition 9.8.** $\dim S_{w_1, w_2, w_3}(\mathrm{Sp}(6, \mathbb{Z})) = O^*(w_1, w_2, w_3) + O(w_1, w_3) \cdot O^*(w_2) + \delta_{w_2 \equiv 0 \bmod 4} \cdot (\delta_{w_2 = w_3 + 2} \cdot S(w_2 - 1) \cdot O^*(w_1) + \delta_{w_1 = w_2 + 2} \cdot S(w_2 + 1) \cdot O^*(w_3)).$

Let us fix a

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{alg}}(\mathrm{Sp}(6))$$

with infinitesimal character $\mathfrak{z}_{\underline{w}}$. We have to determine first whether or not $\Pi(\psi_\infty)$ contains a holomorphic discrete series. Lemma 9.4 ensures that it is the case if and only if for each i such that $\pi_i = 1$ then $d_i = 1$. We thus assume that this property is satisfied and we denote by π the unique element in $\Pi(\psi)$ such that $\pi_\infty \simeq \pi_{\mathrm{hol}}$. We want them to determine $m(\pi)$. By Lemma 9.3 and the remark that follows, we have

$$\tau(\pi)|_{C_\psi} = e_2|_{C_\psi}.$$

For reasons already explained when studying $\mathrm{SO}(8)$, if some n_i is even then $n_i \equiv 0 \bmod 4$, and there is exactly one integer i such that n_i is odd. It follows that either $k = 1$ (stable case) or $k = 2$ and (up to equivalence) $(n_1, n_2) = (4, 3)$. In this latter case $C_\psi = \langle s_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and $\pi_2 \neq 1$.

Case (i) : stable tempered case $\psi = \pi_1 \in \Pi_{\text{alg}}^o(\text{PGL}(7))$. Then ψ_∞ is a discrete series Langlands parameter (hence indeed $\pi_{\text{hol}} \in \Pi(\psi_\infty)$) and $m(\pi) = 1$ by the multiplicity formula. The number of such π is thus $O^*(w_1, w_2, w_3)$.

Case (ii) : endoscopic tempered case, $k = 2$, $d_1 = d_2 = 1$, $\psi = \pi_1 \oplus \pi_2$ where $\pi_1 \in \Pi_{\text{alg}}^o(\text{PGL}(4))$ and $\pi_2 \in \Pi_{\text{alg}}^o(\text{PGL}(3))$. Say π_1 has Hodge numbers $a > b$ and π_2 has Hodge number c . Then again ψ_∞ is a discrete Langlands parameters (hence contains π_{hol}). In particular $\varepsilon_\psi(s_1) = 1$. But $e_2(s_1) = 1$ if and only if $a > c > b$, thus $m(\pi) = 1$ if and only if $a > c > b$. The number of such π is thus $O^*(w_2) \cdot O(w_1, w_3)$.

Case (iii) : endoscopic non-tempered case, $k = 2$, $d_1 = 2$ and $d_2 = 1$, i.e. $\psi = \pi_1[2] \oplus \pi_2$ where $\pi_1 \in \Pi_{\text{alg}}(\text{PGL}(2))$ and $\pi_2 \in \Pi_{\text{alg}}^o(\text{PGL}(3))$. Say π_1 has Hodge number a and π_2 has Hodge number b . This time ψ_∞ is not tempered and

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = (-1)^{1+\text{Max}(a,b)+\frac{a+1}{2}}.$$

But $e_2(s_1) = -1$, so

$$m(\pi) = \begin{cases} 1 & \text{if } b > a \text{ and } a \equiv 3 \pmod{4}, \\ 1 & \text{if } a > b \text{ and } a \equiv 1 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

This concludes the proof of the proposition above.

We remark that we excluded three kinds of parameters thanks to Lemma 9.4, namely $[7]$, $\pi_1 \oplus [3]$ and $\pi_1[2] \oplus [3]$. Alternatively, we could also have argued directly using only Lemma 9.3. Indeed, in those three cases we obviously have $\varepsilon_\psi = 1$, and we see also that $e_2(s_1) = -1$.

10. APPENDIX : ADAMS-JOHNSON PACKETS

10.1. Strong inner forms of compact connected real Lie groups. Let K be a compact connected semisimple Lie group and let G be its complexification. It is a complex semisimple algebraic group equipped with an anti-holomorphic group involution $\sigma : g \mapsto \bar{g}$ such that $K = \{g \in G, \bar{g} = g\}$. As is well-known, K is a maximal compact subgroup of G .

Let T_c be a maximal torus of K and denote by $T \subset G$ the unique maximal torus of G with maximal compact subgroup T_c . Following J. Adams in [Ad], consider the group

$$\mathcal{X}_1(T) = \{t \in T, t^2 \in Z(G)\}.$$

An element of $\mathcal{X}_1(T)$ will be called a *strong inner form* of K (relative to (G, T)). As K is a maximal compact subgroup of G , we have $Z(G) = Z(K) \subset T_c$ and thus $\mathcal{X}_1(T) \subset T_c$. A strong inner form $t \in \mathcal{X}_1(T)$ of K is said *pure* if $t^2 = 1$.

If $t \in \mathcal{X}_1(T)$ we denote by σ_t the group automorphism $\text{Int}(t) \circ \sigma$ of G . We have $\sigma_t^2 = \text{Id}$. It follows that the real linear algebraic Lie group

$$G_t = \{g \in G, \sigma_t(g) = g\}$$

is an inner form of $G_1 = K$ in the usual sense. Observe that $T_c \subset G_t$ and that G_t is stable by σ . The polar decomposition of G relative to K shows then that the group

$$K_t = K \cap G_t,$$

which is also the centralizer of t in K , is a maximal compact subgroup of G_t . The torus T_c is thus a common maximal torus of all the G_t . Any involution of G of the form $\text{Int}(g) \circ \sigma$ with $g \in G$ is actually of the form $\text{Int}(h) \circ \sigma_t \circ \text{Int}(h)^{-1}$ for some $t \in \mathcal{X}_1(T)$ and some $h \in G$ by [Se2, §4.5]. In particular, every inner form of K inside G is G -conjugate to some G_t .

Consider the Weyl group

$$W = W(G, T) = W(K, T_c).$$

It obviously acts on the group $\mathcal{X}_1(T)$, and two strong real forms $t, t' \in \mathcal{X}_1(T)$ are said *equivalent* if they are in a same orbit. If $w \in W$, observe that $\text{Int}(w)$ defines an isomorphism $G_t \rightarrow G_{w(t)}$ which is well-defined up to inner isomorphisms by T_c , so that the group G_t is canonically defined up to inner isomorphisms by the equivalence class of t . This however not the unique kind of redundancy among the groups G_t in general, as for instance $G_t = G_{tz}$ whenever $z \in Z(G)$. We shall denote by $[t] \in W \backslash \mathcal{X}_1(T)$ the equivalence class of $t \in \mathcal{X}_1(T)$ and by $G_{[t]}$ the group G_t "up to inner inner automorphisms". It makes sense in particular to talk about representations of $G_{[t]}$.

As a classical example, consider the case of the even special orthogonal group

$$G = \text{SO}(2r, \mathbb{C}) = \{g \in \text{SL}(2r, \mathbb{C}), {}^t g g = \text{Id}\}$$

with the coordinate-wise complex conjugation σ , i.e. $K = \text{SO}(2r, \mathbb{R})$. Consider the maximal torus

$$T = \text{SO}(2, \mathbb{C})^r \subset G$$

preserving each plane $P_i = \mathbb{C}e_{2i-1} \oplus \mathbb{C}e_{2i}$ for $i = 1, \dots, r$. Here (e_i) is the canonical basis of \mathbb{C}^{2r} . Any $t \in \mathcal{X}_1(T)$ is $W(G, T)$ -equivalent to either a unique element t_j , $0 \leq j \leq r$, where t_j acts by -1 on P_i if $i \leq j$, and by $+1$ otherwise, or to exactly one of the two element $t_{\pm}^* \in T_c$ sending each e_{2i} on $-e_{2i-1}$ for $i < r$ and e_{2r} on $\pm e_{2r-1}$. We have $t_j^2 = 1$ (pure inner forms) and $(t_{\pm}^*)^2 = -1$. We see that

$$K_{t_j} = S(\mathrm{O}(2j) \times \mathrm{O}(2r - 2j))$$

and $G_{t_j} \simeq \mathrm{SO}(2j, 2r - 2j)$. In particular, $G_{t_j} \simeq G_{t_{j'}}$ if and only if $j = j'$ or $j + j' = r$. Moreover, $K_{t_{\pm}^*}$ is isomorphic to the unitary group in r variables and $G_{t_{\pm}^*}$ is the Lie group sometimes denoted by $\mathrm{SO}(2r)^*$. Observe that the only quasi-split group among the G_{t_j} and $G_{t_{\pm}^*}$ is $\mathrm{SO}(r + 1, r - 1)$ if r is odd, $\mathrm{SO}(r, r)$ if r is even. In particular, the split group $\mathrm{SO}(r, r)$ is a pure inner form of K if and only if r is even.

We leave as an exercise to the reader to treat the similar cases $G = \mathrm{Sp}(2g, \mathbb{C})$ and $G = \mathrm{SO}(2r + 1, \mathbb{C})$ which are only easier. When $G = \mathrm{Sp}(2g, \mathbb{C})$, each twisted form of K is actually inner as $\mathrm{Out}(G) = 1$. In this case the (inner) split form $\mathrm{Sp}(2g, \mathbb{R})$ is not a pure inner form of K , it corresponds to the single equivalence class of t such that $t^2 = -1$. When $G = \mathrm{SO}(2r + 1, \mathbb{C})$, then $Z(G) = \mathrm{Out}(G) = 1$, and the equivalence classes of strong inner forms of K are in bijection with the isomorphism classes of inner forms of K , namely the real special orthogonal groups $\mathrm{SO}(2j, 2r + 1 - 2j)$ of signature $(2j, 2r + 1 - 2j)$ for $j = 0, \dots, r$.

10.2. Adams-Johnson parameters. We refer to Kottwitz' exposition in [K2, p. 195] and to Adams paper [Ad], of which the presentation below is very much inspired.

We keep the assumptions of §10.1 and we assume from now on that the set of strong real forms of K contains a split real group. It is equivalent to ask that the center of the simply connected covering G_{sc} of G is an elementary abelian 2-group, i.e. G has no factor of type E_6 , or type A_n or D_{2n-1} for $n > 1$. We may view the Langlands dual group of G as a complex connected semisimple algebraic group \widehat{G} , omitting the trivial Galois action. We fix once and for all a Borel subgroup B containing T , which gives rise to a dual Borel pair $(\widehat{T}, \widehat{B})$ in \widehat{G} .

Denote by $\Psi(G)$ the set of Arthur parameters of the inner forms of K . This is the set of continuous homomorphisms

$$W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow \widehat{G}$$

which are \mathbb{C} -algebraic on the $\mathrm{SL}(2, \mathbb{C})$ -factor and such that the image of any element of $W_{\mathbb{R}}$ is semisimple. Two such parameters are said equivalent if they are conjugate under \widehat{G} . Fix $\psi \in \Psi(G)$. Let \widehat{L} be the centralizer in \widehat{G} of $\psi(W_{\mathbb{C}})$, which is a Levi subgroup of some parabolic subgroup of \widehat{G} ; as $W_{\mathbb{C}}$ is commutative

$$\psi(W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})) \subset \widehat{L}.$$

Let us denote by C_{ψ} the centralizer of $\mathrm{Im}(\psi)$ in \widehat{G} and consider the following two properties of a $\psi \in \Psi(G)$.

(a) $\psi(\mathrm{SL}_2(\mathbb{C}))$ contains a regular unipotent element of \widehat{L} .

(b) C_ψ is finite.

Property (a) forces in particular the centralizer of $\psi(\mathrm{SL}_2(\mathbb{C}))$ in \widehat{L} to be $Z(\widehat{L})$, thus under (a) we have

$$C_\psi = Z(\widehat{L})^\theta$$

where $\theta = \mathrm{Int}(\psi(j))$. Moreover, if one assumes (a) then property (b) is equivalent to the assertion that the involution θ acts as the inversion on $Z(\widehat{L})^0$. If A is an abelian group, we denote by $A[2]$ the subgroup of elements $a \in A$ such that $a^2 = 1$.

Lemma 10.3. *If $\psi \in \Psi(G)$ satisfies (a) and (b) then $C_\psi = Z(\widehat{L})[2]$.*

Proof — Indeed, as a general fact one has $Z(\widehat{L}) = Z(\widehat{G})Z(\widehat{L})^0$, because the character group of the diagonalisable group $Z(\widehat{L})/Z(\widehat{G})$ is free, being the quotient of the root lattice of \widehat{G} by the root lattice of \widehat{L} . We also obviously have $Z(\widehat{G}) \subset C_\psi$ (θ acts trivially on $Z(\widehat{G})$), thus $C_\psi = Z(\widehat{G})(Z(\widehat{L})^0)^\theta$. By assumption on G one has $Z(\widehat{G}) = Z(\widehat{G})[2]$. As θ acts as the inversion on $Z(\widehat{L})^0$ one obtains $Z(\widehat{L})[2] = Z(\widehat{G})(Z(\widehat{L})^0[2]) = C_\psi$. \square \square

To any $\psi \in \Psi(G)$ one may attach following Arthur a Langlands parameter

$$\varphi_\psi : W_{\mathbb{R}} \rightarrow \widehat{G}$$

defined by restricting ψ along the homomorphism

$$W_{\mathbb{R}} \rightarrow W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$$

which is the identity on the first factor and the representation $|\cdot|^{1/2} \oplus |\cdot|^{-1/2}$ on the second factor. Here, $|\cdot| : W_{\mathbb{R}} \rightarrow \mathbb{R}_{>0}$ is the norm homomorphism, sending j to 1 and $z \in W_{\mathbb{C}}$ to $z\bar{z}$. Up to \widehat{G} -conjugation of ψ , we may assume that

$$\varphi_\psi(W_{\mathbb{C}}) \subset \widehat{T}.$$

We follow Langlands notation⁹ and write

$$(10) \quad \varphi_\psi(z) = z^\lambda \bar{z}^\mu$$

for any $z \in W_{\mathbb{C}}$, where $\lambda, \mu \in X_*(\widehat{T}) \otimes \mathbb{C} = \mathrm{Lie}_{\mathbb{C}}(\widehat{T})$ and $\lambda - \mu \in X_*(\widehat{T})$. The \widehat{G} -conjugacy class of λ in \mathfrak{g} is called the infinitesimal character of ψ (and φ_ψ) and will be denoted by z_ψ . The last condition we shall consider is :

(c) z_ψ is the infinitesimal character of a finite dimensional \mathbb{C} -algebraic representation of G .

Under assumption (c), it follows that \widehat{T} is the centralizer of $\varphi_\psi(W_{\mathbb{C}})$ in \widehat{G} , and up to conjugating ψ by the Weyl group of $(\widehat{G}, \widehat{T})$ one may assume that λ is dominant with respect to \widehat{B} .

⁹If $z \in \mathbb{C}^*$, and if $a, b \in \mathbb{C}$ satisfy $a - b \in \mathbb{Z}$, set $z^a \bar{z}^b = e^{ax + b\bar{x}}$ where $x \in \mathbb{C}$ is any element such that $z = e^x$.

Definition 10.4. *The subset of $\psi \in \Psi(G)$ satisfying (a), (b) and (c) will be denoted by $\Psi_{AJ}(G)$.*

As an example, consider the group $G = \mathrm{Sp}(2g, \mathbb{C})$, so that $\widehat{G} = \mathrm{SO}(2g + 1, \mathbb{C})$. Let $\mathrm{St} : \widehat{G} \rightarrow \mathrm{GL}(2g + 1, \mathbb{C})$ be the standard representation of \widehat{G} . Let $\psi \in \Psi(G)$. Then $\psi \in \Psi_{AJ}(G)$ if and only if

$$\mathrm{St} \circ \psi \simeq \varepsilon^s \otimes \nu_{d_0} \oplus \bigoplus_{i \neq 0} \mathrm{I}_{w_i} \otimes \nu_{d_i}$$

for some positive integers w_i and d_i with $(-1)^{w_i+d_i-1} = 1$ for each i , with the convention $w_0 = 0$ and where $w_i > 0$ if $i \neq 0$, such that the $2g + 1$ even integers

$$\pm w_i + d_i - 1, \pm w_i + d_i - 3, \dots, \pm w_i - d_i + 1$$

are distinct. The integer s is congruent mod 2 to the number of $i \neq 0$ such that w_i is even. Moreover, the equivalence class of ψ is uniquely determined by the isomorphism class of $\mathrm{St} \circ \psi$. If ψ is as above, then $\widehat{L} \simeq \mathrm{SO}(d_0, \mathbb{C}) \times \prod_{i \neq 0} \mathrm{GL}(d_i, \mathbb{C})$ and $\mathrm{C}_\psi \simeq \prod_{i \neq 0} \{\pm 1\}$.

The case $G = \mathrm{SO}(2r + 1, \mathbb{C})$ is quite similar, one simply has to replace the condition $(-1)^{w_i+d_i-1} = 1$ by $(-1)^{w_i+d_i-1} = -1$, and there is no more restriction on $s \bmod 2$. The case $G = \widehat{G} = \mathrm{SO}(2r, \mathbb{C})$ is slightly different but left as an exercise to the reader.

10.5. Adams-Johnson packets. In the paper [AdJ], J. Adams and J. Johnson associate to any $\psi \in \Psi_{AJ}(G)$, and to any equivalence class of strong inner forms $t \in \mathcal{X}_1(T)$, a finite set of (usually non-tempered) irreducible unitary representations of $G_{[t]}$ satisfying certain predictions of Arthur. More precisely to any $t \in \mathcal{X}_1(T)$ they associate an isomorphism class

$$\pi_t(\lambda)$$

of unitary irreducible representations of G_t , where λ is as in (10). Let us recall briefly their definition.

Let \widehat{P} be the parabolic subgroup of \widehat{G} containing \widehat{B} and with Levi subgroup \widehat{L} , let $P \subset G$ be parabolic subgroup dual to \widehat{P} , and let $L \subset P$ be the Levi subgroup dual to \widehat{L} . Concretely, as

$$\psi(W_{\mathbb{C}}) \subset Z(\widehat{L})^0 \subset \widehat{T}$$

there is a unique $\lambda_0 \in X_*(\widehat{T})[1/2]$ such that $\psi(z) = (z/\bar{z})^{\lambda_0}$ for all $z \in W_{\mathbb{C}}$, and L is the stabilizer in G of $\lambda_0 \in \mathfrak{g}^*$. Of course we have $T \subset L$. As $\sigma_t(\lambda_0) = -\lambda_0$, it follows that σ_t preserves L , and thus the real Lie group

$$L_t = G_t \cap L$$

is a real form of L containing T_c . The real group L_t is even an inner form of $K \cap L$ (a maximal compact subgroup of L). Moreover, the Cartan involution $\mathrm{Int}(t)$ of G_t preserves P as $T \subset P$, and defines a Cartan involution of L_t as well. Assume that L_t is connected to simplify (see loc. cit. for the general case). There is a unique one-dimensional unitary character χ_λ of L_t whose restriction to T_c is $\lambda - \rho$ where ρ denotes the half-sum of the positive roots of (G, T) with respect to B . Adams and Johnson define $\pi_t(\lambda)$ as the

cohomological induction relative to P from the $(\mathfrak{l}, K_t \cap L_t)$ -module χ_λ to (\mathfrak{g}, K_t) . To emphasize the dependence on P in this construction, we shall sometimes write

$$\pi_{t,P}(\lambda)$$

rather than $\pi_t(\lambda)$.

The isomorphisms $\text{Int}(w) : G_t \rightarrow G_{w(t)}$, for $w \in W$, allow to consider the collection of representations $\pi_{w(t)}(\lambda)$ as representations of $G_{[t]}$. The set of such representations is the Adams-Johnson packet of $G_{[t]}$ attached to ψ , and we shall denote it by

$$\Pi(\psi, G_{[t]}).$$

It turns out that for $t, t' \in \mathcal{X}_1(T)$ in a same W -orbit, then $\pi_t(\lambda) \simeq \pi_{t'}(\lambda)$ if and only if t and t' are in a same $W(L, T)$ -orbit. Observe also that for $t \in \mathcal{X}_1(T)$ we have

$$\{w \in W, w(t) = t\} = W(K_t, T_c).$$

It follows that $\Pi(\psi, G_{[t]})$ is in natural bijection with $W(L, T) \backslash W / W(K_t, T_c)$ and in particular that $|\Pi(\psi, G_{[t]})|$ is the number of such double cosets.

Lemma 10.6. *The representation $\pi_{t,P}(\lambda)$ is a discrete series representation if and only if $t \in Z(L)$.*

Proof — Indeed, as recalled loc. cit., $\pi_{t,P}(\lambda)$ is a discrete series representation if and only if L_t is compact. The result follows as $\text{Int}(t)$ is a Cartan involution of L_t . Note that for such a t the group L_t is of course always connected as so is L . \square

In the special case $t \in Z(G)$, i.e. G_t is compact, it follows that $\Pi(\psi, G_{[t]})$ is the singleton made of the unique irreducible representations of highest weight $\lambda - \rho$ relative to B . A more important special case is the one with $\psi(\text{SL}(2, \mathbb{C})) = \{1\}$. In this case ψ is nothing more than a discrete series parameter in the sense of Langlands. Here (a) is automatic, (b) implies (c), $\varphi_\psi = \psi$ and $\widehat{L} = \widehat{T}$. Then $\pi_\lambda(t)$ is the discrete series representation with Harish-Chandra parameter λ , and $\Pi(\psi, G_{[t]})$ is simply the set of isomorphism classes of discrete series representations of $G_{[t]}$ with infinitesimal character z_ψ .

10.7. Shelstad's parameterization map. What follows is again much inspired from [K2, p. 195] and [Ad]. We fix a $\psi \in \Psi_{\text{AJ}}(G)$ and keep the assumptions and notations of the previous paragraphs. We denote by

$$\widetilde{\Pi}(\psi)$$

the disjoint union of the sets $\Pi(\psi, G_{[t]})$ where $[t]$ runs over the equivalence classes of strong real forms of K . As already explained in the previous paragraph, the map $\mathcal{X}_1(T) \rightarrow \widetilde{\Pi}(\psi), t \mapsto \pi_t(\lambda)$, induces a bijection

$$(11) \quad W(L, T) \backslash \mathcal{X}_1(T) \xrightarrow{\sim} \widetilde{\Pi}(\psi).$$

Define S_ψ as the inverse image of C_ψ under the simply connected covering

$$p : \widehat{G}_{\text{sc}} \rightarrow \widehat{G}.$$

Following [Sh1], [Sh2], Langlands, Arthur, [AdJ], [ABV] and [K2], the set $\tilde{\Pi}(\psi)$ is equipped with a natural map

$$\tau_0 : \tilde{\Pi}(\psi) \rightarrow \text{Hom}(S_\psi, \mathbb{C}^*)$$

that we shall now describe in the style of Adams in [Ad]. Observe first that S_ψ is the inverse image of C_ψ in $\hat{T}_{\text{sc}} = p^{-1}(\hat{T})$, hence it is an abelian group.

Lemma 10.8. $S_\psi \subset (p^{-1}(\hat{T}[2]))^{W(L,T)}$.

Proof — By Lemma 10.3 and the inclusion $C_\psi \subset C_{\varphi_\psi} = \hat{T}$ described in §10.2, one obtains a canonical inclusion $C_\psi \subset \hat{T}[2]$. Moreover, $\hat{L}_{\text{sc}} := p^{-1}(\hat{L})$ is a Levi subgroup of \hat{G} containing \hat{T}_{sc} and thus $p^{-1}(Z(\hat{L})) = Z(\hat{L}_{\text{sc}})$ and $W(L, T) = W(\hat{L}, \hat{T}) = W(\hat{L}_{\text{sc}}, \hat{T}_{\text{sc}})$. In particular, $W(L, T)$ acts trivially on $p^{-1}(Z(\hat{L}))$, hence trivially on S_ψ . \square

On the other hand, there is a natural perfect W -equivariant pairing

$$\mathcal{X}_1(T) \times p^{-1}(\hat{T}[2]) \rightarrow \mathbb{C}^*.$$

Indeed, if $P^\vee(T)$ denotes the co-weight lattice of T we have natural identifications

$$\mathcal{X}_1(T) = \frac{1}{2}P^\vee(T)/X_*(T) \quad \text{and} \quad p^{-1}(\hat{T}[2]) = \frac{1}{2}X_*(\hat{T})/X_*(\hat{T}_{\text{sc}})$$

via $\mu \mapsto e^{2i\pi\mu}$. The pairing alluded above is then $(\mu, \mu') \mapsto e^{i\pi\langle \mu, \mu' \rangle}$, where \langle, \rangle is the canonical perfect pairing $X_*(T) \otimes \mathbb{Q} \times X_*(\hat{T}) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$. The resulting pairing is perfect as well as $X_*(\hat{T}_{\text{sc}})$ is canonically identified by \langle, \rangle with the root lattice of T .

One then defines τ_0 as follows. Fix $\pi \in \tilde{\Pi}(\psi)$. By the bijection (11), there is an element $t \in \mathcal{X}_1(T)$, whose $W(L, T)$ -orbit is canonically defined, such that $\pi \simeq \pi_t(\lambda)$. The perfect pairing above associates to t a unique character $p^{-1}(\hat{T}[2]) \rightarrow \mathbb{C}^*$, whose restriction to S_ψ only depends on the $W(L, T) = W(\hat{L}, \hat{T})$ -orbit of t by Lemma 10.8 : define $\tau_0(\pi)$ as this character of S_ψ .

This parameterization is discussed in details in [Ad] in the discrete series case, i.e. when $\hat{L} = \hat{T}$. It follows from the previous discussion that τ_0 is a bijection in this case, as $S_\psi = p^{-1}(\hat{T}[2])$. The following simple lemma shows that the determination of the parameterization of discrete series in $\tilde{\Pi}(\psi)$ for general ψ reduces to this latter case.

Observe following [K2] that for any $\psi \in \Psi_{\text{AJ}}(G)$ there is a unique discrete series parameter $\psi_{\text{disc}} \in \Psi_{\text{AJ}}(G)$ such that the centralizers of $\varphi_\psi(W_{\mathbb{C}})$ and $\psi_{\text{disc}}(W_{\mathbb{C}})$ coincide, and such that the parameters λ for ψ and ψ_{disc} defined by (10) coincide as well. In particular, ψ and ψ_{disc} have the same infinitesimal character. Of course, $\psi_{\text{disc}} \neq \varphi_\psi$ if $\psi \neq \psi_{\text{disc}}$. If ψ is normalized as before, we have canonical inclusions

$$C_\psi \subset C_{\psi_{\text{disc}}} = \hat{T}[2] \quad \text{and} \quad S_\psi \subset S_{\psi_{\text{disc}}} = p^{-1}(\hat{T}[2]).$$

The discrete series representations belonging to $\tilde{\Pi}(\psi)$ are exactly the elements of $\tilde{\Pi}(\psi) \cap \tilde{\Pi}(\psi_{\text{disc}})$. It will be important to distinguish in the next lemma the parameterization maps

τ_0 of $\tilde{\Pi}(\psi)$ and $\tilde{\Pi}(\psi_{\text{disc}})$, so we shall denote them respectively by $\tau_{0,\psi}$ and $\tau_{0,\psi_{\text{disc}}}$. Recall from Lemma 10.6 that $\pi_t(\lambda)$ is a discrete series representation if and only if $t \in Z(L)$. The following lemma is a variant of an observation by Kottwitz in [K2].

Lemma 10.9. *Let $\psi \in \Psi_{\text{AJ}}(G)$ and let $\pi \in \tilde{\Pi}(\psi) \cap \tilde{\Pi}(\psi_{\text{disc}})$. Then*

$$\tau_{0,\psi}(\pi) = \tau_{0,\psi_{\text{disc}}}(\pi)|_{S_\psi}.$$

Proof — We have $\pi \simeq \pi_{t,B}(\lambda) \in \tilde{\Pi}(\psi_{\text{disc}})$ for a unique $t \in \mathcal{X}_1(T)$ and we also have $\pi \simeq \pi_{t',P}(\lambda) \in \tilde{\Pi}(\psi)$ for a unique element $t' \in \mathcal{X}_1(T) \cap Z(L)$ by Lemma 10.6 (note that t' is fixed by $W(L, T)$). Applying the "transitivity of cohomological induction" via the compact connected group L_t (use e.g. [KnV, Cor. 11.86 (b)], here $q_0 = 0$), we have $\pi_{t,B}(\lambda) \simeq \pi_{t',P}(\lambda)$. It follows that $t = t'$, which concludes the proof. \square

Observe that for $t \in \mathcal{X}_1(T)$, G_t is compact if and only if $t \in Z(G)$, in which case it coincides with its equivalence class (and it is fixed by $W(L, T)$). The associated representation $\pi_{t,P}(\lambda)$ is the unique finite dimensional representation of G_t with infinitesimal character z_ψ . It occurs $|Z(G)|$ times in $\tilde{\Pi}(\psi)$, once for each $t \in Z(G)$, and these representations are perhaps the most obvious elements in $\tilde{\Pi}(\psi) \cap \tilde{\Pi}(\psi_{\text{disc}})$. To understand their characters we have to describe the image

$$\mathcal{N}(T)$$

of $Z(G)$ under the homomorphism $\mathcal{X}_1(T) \rightarrow \text{Hom}(p^{-1}(\hat{T}[2]), \mathbb{C}^*)$ induced by the canonical pairing. Observe that $Z(G) = \{t^2, t \in \mathcal{X}_1(T)\}$. The following lemma follows.

Lemma 10.10. *The subgroup $\mathcal{N}(T) \subset \text{Hom}(p^{-1}(\hat{T}[2]), \mathbb{C}^*)$ is the subgroup of squares, or equivalently of characters which are trivial on $\hat{T}_{\text{sc}}[2]$.*

The parameterization τ_0 of $\tilde{\Pi}(\psi)$ introduced so far is the one we shall need up to a translation by a certain character b_ψ of S_ψ (or "base point of ψ "). Write again temporarily $\tau_{0,\psi}$ for τ_0 in order to emphasize its dependence on ψ and we write character groups additively. The map

$$\tau_\psi = \tau_{0,\psi} - b_\psi$$

has to satisfy the following two conditions :

- (i) Lemma 10.9 holds with $\tau_{0,\psi}$ and $\tau_{0,\psi_{\text{disc}}}$ replaced respectively by τ_ψ and $\tau_{\psi_{\text{disc}}}$.
- (ii) If ψ is a discrete series parameter, and if $\pi = \pi_t(\lambda) \in \tilde{\Pi}(\psi)$ satisfies $\tau_\psi(\pi) = 1$, i.e. $\tau_{0,\psi}(\pi) = b_\psi$, then G_t is a split real group and π is generic with respect to some Whittaker functional.

Normalize ψ as in §10.2. Following [Ad], consider the element

$$t_b = e^{i\pi\rho^\vee} \in \mathcal{X}(T)$$

where $\rho^\vee \in X_*(T)$ is the half-sum of the positive coroots with respect to (G, B, T) . Under the identification $\mathcal{X}_1(T) = \frac{1}{2}P^\vee(T)/X_*(T)$, t_b is the class of $\frac{1}{2}\rho^\vee$. In particular, under the canonical pairing between $\hat{S}_{\psi_{\text{disc}}}$ and $\mathcal{X}_1(T)$ the element t_b corresponds to the restriction

to $S_{\psi_{\text{disc}}}$ of the character $\rho^\vee \in X^*(\widehat{T}_{\text{sc}})$. The characteristic property of t_b is that for any t in the coset $Z(G)t_b \subset \mathcal{X}_1(T)$, then $\pi_t(\lambda)$ is a generic (or "large" in the sense of Vogan) discrete series of the split group G_t . To fulfill the conditions (i) and (ii) one simply set $b_\psi = \rho^\vee$.

Definition 10.11. *If $\psi \in \Psi_{\text{AJ}}(G)$, the canonical parameterization*

$$\tau : \widetilde{\Pi}(\psi) \rightarrow \text{Hom}(S_\psi, \mathbb{C}^*)$$

is defined by $\tau = \tau_0 - \rho^\vee|_{S_\psi}$ where \widehat{T} is the centralizer of $\varphi_\psi(W_{\mathbb{C}})$, \widehat{B} is the unique Borel subgroup of \widehat{G} containing \widehat{T} with respect to which the element λ defined by (10) is dominant, and ρ^\vee is the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$.

Corollary 10.12. *If $\pi \in \widetilde{\Pi}(\psi)$ is a finite dimensional representation, then $\tau(\pi) \in \mathcal{N}(T) - \rho^\vee$.*

We end this paragraph by collecting a couple of well-known and simple facts we used in the paper. For $t, t' \in \mathcal{X}_1(T)$, G_t and $G_{t'}$ are pure inner forms if and only if $t^2 = (t')^2$.

Corollary 10.13. *K is a pure inner form of a split group if and only if $\rho^\vee \in X^*(\widehat{T})$.*

Indeed, G_t is a pure inner form of a split group if and only if $t^2 = t_b^2 = (-1)^{2\rho^\vee}$.

Corollary 10.14. *Let $t \in \mathcal{X}_1(T)$. Then G_t is a pure inner form of a split group if and only if the character of $p^{-1}(\widehat{T}[2])$ associated to tt_b^{-1} under the canonical pairing factors through $\widehat{T}[2]$.*

11. TABLES

$$G = \mathrm{SO}(7, \mathbb{R}), \Gamma = W^+(\mathrm{E}_7).$$

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0, 0, 0)	1	(9, 6, 3)	2	(10, 7, 2)	1	(10, 10, 10)	2	(11, 9, 0)	2
(4, 4, 4)	1	(9, 6, 4)	1	(10, 7, 3)	3	(11, 3, 0)	1	(11, 9, 1)	1
(6, 0, 0)	1	(9, 6, 6)	1	(10, 7, 4)	2	(11, 3, 2)	1	(11, 9, 2)	4
(6, 4, 0)	1	(9, 7, 2)	1	(10, 7, 5)	2	(11, 4, 1)	1	(11, 9, 3)	4
(6, 6, 0)	1	(9, 7, 3)	1	(10, 7, 6)	2	(11, 4, 3)	2	(11, 9, 4)	5
(6, 6, 6)	1	(9, 7, 4)	2	(10, 7, 7)	1	(11, 4, 4)	1	(11, 9, 5)	4
(7, 4, 3)	1	(9, 7, 6)	1	(10, 8, 0)	3	(11, 5, 0)	2	(11, 9, 6)	5
(7, 6, 3)	1	(9, 8, 1)	1	(10, 8, 2)	3	(11, 5, 2)	2	(11, 9, 7)	3
(7, 7, 3)	1	(9, 8, 3)	1	(10, 8, 3)	1	(11, 5, 3)	1	(11, 9, 8)	2
(7, 7, 7)	1	(9, 8, 4)	1	(10, 8, 4)	4	(11, 5, 4)	1	(11, 9, 9)	1
(8, 0, 0)	1	(9, 8, 5)	1	(10, 8, 5)	1	(11, 6, 1)	2	(11, 10, 1)	3
(8, 4, 0)	1	(9, 8, 6)	1	(10, 8, 6)	3	(11, 6, 2)	1	(11, 10, 2)	3
(8, 4, 2)	1	(9, 9, 0)	1	(10, 8, 7)	1	(11, 6, 3)	4	(11, 10, 3)	5
(8, 4, 4)	1	(9, 9, 3)	1	(10, 8, 8)	1	(11, 6, 4)	2	(11, 10, 4)	4
(8, 6, 0)	1	(9, 9, 4)	1	(10, 9, 1)	2	(11, 6, 5)	2	(11, 10, 5)	6
(8, 6, 2)	1	(9, 9, 6)	1	(10, 9, 2)	1	(11, 6, 6)	2	(11, 10, 6)	5
(8, 6, 4)	1	(9, 9, 9)	1	(10, 9, 3)	3	(11, 7, 0)	1	(11, 10, 7)	5
(8, 6, 6)	1	(10, 0, 0)	1	(10, 9, 4)	2	(11, 7, 1)	1	(11, 10, 8)	3
(8, 7, 2)	1	(10, 2, 0)	1	(10, 9, 5)	3	(11, 7, 2)	4	(11, 10, 9)	2
(8, 7, 4)	1	(10, 4, 0)	2	(10, 9, 6)	2	(11, 7, 3)	3	(11, 10, 10)	2
(8, 7, 6)	1	(10, 4, 2)	1	(10, 9, 7)	2	(11, 7, 4)	4	(11, 11, 1)	1
(8, 8, 0)	1	(10, 4, 3)	1	(10, 9, 8)	1	(11, 7, 5)	3	(11, 11, 2)	2
(8, 8, 2)	1	(10, 4, 4)	2	(10, 9, 9)	1	(11, 7, 6)	3	(11, 11, 3)	3
(8, 8, 4)	1	(10, 5, 1)	1	(10, 10, 0)	2	(11, 7, 7)	2	(11, 11, 4)	2
(8, 8, 6)	1	(10, 5, 3)	1	(10, 10, 2)	2	(11, 8, 1)	3	(11, 11, 5)	3
(8, 8, 8)	1	(10, 6, 0)	2	(10, 10, 3)	2	(11, 8, 2)	2	(11, 11, 6)	3
(9, 3, 0)	1	(10, 6, 2)	2	(10, 10, 4)	4	(11, 8, 3)	5	(11, 11, 7)	3
(9, 4, 3)	1	(10, 6, 3)	1	(10, 10, 5)	2	(11, 8, 4)	4	(11, 11, 8)	2
(9, 4, 4)	1	(10, 6, 4)	3	(10, 10, 6)	4	(11, 8, 5)	5	(11, 11, 9)	1
(9, 5, 0)	1	(10, 6, 5)	1	(10, 10, 7)	2	(11, 8, 6)	4	(11, 11, 10)	1
(9, 5, 2)	1	(10, 6, 6)	2	(10, 10, 8)	2	(11, 8, 7)	3	(11, 11, 11)	1
(9, 6, 1)	1	(10, 7, 1)	2	(10, 10, 9)	2	(11, 8, 8)	1	(12, 0, 0)	2

TABLE 2. The nonzero $d(\lambda) = \dim V_\lambda^\Gamma$ for $\lambda = (n_1, n_2, n_3)$ with $n_1 \leq 11$.

$$G = \mathrm{SO}(8, \mathbb{R}), \Gamma = W^+(\mathrm{E}_8).$$

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0, 0, 0, 0)	1	(10, 9, 1, 0)	1	(11, 8, 5, 2)	1	(11, 11, 3, 3)	2
(4, 4, 4, 4)	1	(10, 9, 4, 3)	1	(11, 8, 6, 1)	1	(11, 11, 4, 4)	1
(6, 6, 0, 0)	1	(10, 9, 5, 0)	1	(11, 8, 6, 3)	1	(11, 11, 5, 1)	1
(6, 6, 6, 6)	1	(10, 9, 6, 1)	1	(11, 8, 7, 2)	1	(11, 11, 5, 5)	2
(7, 7, 3, 3)	1	(10, 9, 7, 0)	1	(11, 8, 7, 4)	1	(11, 11, 6, 2)	1
(7, 7, 7, 7)	1	(10, 9, 9, 2)	1	(11, 8, 8, 3)	1	(11, 11, 6, 6)	2
(8, 0, 0, 0)	1	(10, 10, 0, 0)	1	(11, 9, 2, 0)	1	(11, 11, 7, 1)	2
(8, 4, 4, 0)	1	(10, 10, 2, 2)	1	(11, 9, 3, 1)	1	(11, 11, 7, 7)	2
(8, 6, 6, 0)	1	(10, 10, 3, 3)	1	(11, 9, 4, 2)	1	(11, 11, 8, 0)	2
(8, 7, 7, 0)	1	(10, 10, 4, 0)	1	(11, 9, 5, 1)	1	(11, 11, 8, 4)	1
(8, 8, 0, 0)	1	(10, 10, 4, 4)	2	(11, 9, 5, 3)	1	(11, 11, 8, 8)	1
(8, 8, 2, 2)	1	(10, 10, 5, 5)	1	(11, 9, 6, 0)	2	(11, 11, 9, 3)	1
(8, 8, 4, 4)	1	(10, 10, 6, 0)	1	(11, 9, 6, 4)	1	(11, 11, 9, 9)	1
(8, 8, 6, 6)	1	(10, 10, 6, 2)	1	(11, 9, 7, 1)	1	(11, 11, 10, 2)	1
(8, 8, 8, 0)	1	(10, 10, 6, 6)	2	(11, 9, 7, 3)	1	(11, 11, 10, 10)	1
(8, 8, 8, 8)	1	(10, 10, 7, 1)	1	(11, 9, 7, 5)	1	(11, 11, 11, 3)	1
(9, 6, 3, 0)	1	(10, 10, 7, 7)	1	(11, 9, 8, 2)	1	(11, 11, 11, 11)	1
(9, 7, 4, 2)	1	(10, 10, 8, 0)	1	(11, 9, 9, 3)	1	(12, 0, 0, 0)	1
(9, 8, 6, 1)	1	(10, 10, 8, 4)	1	(11, 10, 1, 0)	1	(12, 4, 0, 0)	1
(9, 9, 3, 3)	1	(10, 10, 8, 8)	1	(11, 10, 3, 2)	1	(12, 4, 4, 0)	1
(9, 9, 4, 4)	1	(10, 10, 9, 3)	1	(11, 10, 4, 1)	1	(12, 4, 4, 4)	1
(9, 9, 6, 6)	1	(10, 10, 9, 9)	1	(11, 10, 4, 3)	1	(12, 5, 3, 2)	1
(9, 9, 9, 9)	1	(10, 10, 10, 2)	2	(11, 10, 5, 0)	1	(12, 6, 0, 0)	1
(10, 4, 0, 0)	1	(10, 10, 10, 10)	2	(11, 10, 5, 2)	1	(12, 6, 2, 0)	1
(10, 4, 4, 2)	1	(11, 4, 4, 3)	1	(11, 10, 5, 4)	1	(12, 6, 4, 0)	1
(10, 6, 0, 0)	1	(11, 5, 2, 0)	1	(11, 10, 6, 1)	2	(12, 6, 4, 2)	1
(10, 6, 4, 0)	1	(11, 6, 3, 0)	1	(11, 10, 6, 5)	1	(12, 6, 6, 0)	2
(10, 6, 6, 2)	1	(11, 6, 4, 3)	1	(11, 10, 7, 0)	3	(12, 6, 6, 4)	1
(10, 7, 4, 1)	1	(11, 6, 6, 3)	1	(11, 10, 7, 2)	1	(12, 7, 3, 0)	1
(10, 7, 6, 3)	1	(11, 7, 3, 1)	1	(11, 10, 7, 6)	1	(12, 7, 3, 2)	1
(10, 7, 7, 2)	1	(11, 7, 4, 0)	1	(11, 10, 8, 1)	1	(12, 7, 4, 1)	1
(10, 8, 2, 0)	1	(11, 7, 5, 1)	1	(11, 10, 8, 3)	1	(12, 7, 4, 3)	1
(10, 8, 4, 0)	1	(11, 7, 6, 2)	1	(11, 10, 9, 2)	1	(12, 7, 5, 2)	1
(10, 8, 4, 2)	1	(11, 7, 7, 3)	1	(11, 10, 9, 4)	1	(12, 7, 6, 1)	1
(10, 8, 6, 0)	1	(11, 8, 3, 0)	1	(11, 10, 10, 3)	2	(12, 7, 6, 3)	1
(10, 8, 6, 4)	1	(11, 8, 4, 1)	1	(11, 11, 1, 1)	1	(12, 7, 7, 0)	1
(10, 8, 8, 2)	1	(11, 8, 5, 0)	1	(11, 11, 2, 2)	1	(12, 7, 7, 4)	1

TABLE 3. The nonzero $d(\lambda) = \dim V_\lambda^\Gamma$ for $\lambda = (n_1, n_2, n_3, n_4)$ with $n_1 \leq 11$.

$$G = \mathrm{SO}(9, \mathbb{R}), \Gamma = \mathrm{W}(\mathrm{E}_8).$$

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0, 0, 0, 0)	1	(8, 6, 6, 2)	1	(9, 3, 0, 0)	1	(9, 7, 7, 3)	2	(9, 8, 8, 8)	1
(2, 0, 0, 0)	1	(8, 6, 6, 4)	1	(9, 4, 4, 1)	1	(9, 7, 7, 4)	2	(9, 9, 3, 1)	1
(4, 0, 0, 0)	1	(8, 6, 6, 6)	2	(9, 4, 4, 3)	1	(9, 7, 7, 6)	1	(9, 9, 3, 3)	2
(4, 4, 4, 4)	1	(8, 7, 3, 3)	1	(9, 4, 4, 4)	1	(9, 7, 7, 7)	1	(9, 9, 4, 0)	1
(5, 4, 4, 4)	1	(8, 7, 4, 1)	1	(9, 5, 0, 0)	1	(9, 8, 1, 0)	1	(9, 9, 4, 2)	2
(6, 0, 0, 0)	1	(8, 7, 4, 3)	2	(9, 5, 4, 0)	1	(9, 8, 2, 2)	1	(9, 9, 4, 3)	2
(6, 4, 4, 4)	1	(8, 7, 5, 3)	1	(9, 5, 4, 2)	1	(9, 8, 3, 0)	2	(9, 9, 4, 4)	2
(6, 6, 0, 0)	1	(8, 7, 6, 1)	1	(9, 5, 4, 4)	1	(9, 8, 3, 2)	2	(9, 9, 5, 1)	1
(6, 6, 2, 0)	1	(8, 7, 6, 3)	2	(9, 6, 1, 0)	1	(9, 8, 4, 1)	2	(9, 9, 5, 2)	1
(6, 6, 4, 0)	1	(8, 7, 6, 5)	1	(9, 6, 3, 0)	2	(9, 8, 4, 2)	2	(9, 9, 5, 3)	3
(6, 6, 6, 0)	1	(8, 7, 7, 1)	1	(9, 6, 3, 2)	1	(9, 8, 4, 3)	3	(9, 9, 5, 4)	2
(6, 6, 6, 6)	1	(8, 7, 7, 3)	2	(9, 6, 4, 1)	2	(9, 8, 4, 4)	2	(9, 9, 6, 0)	1
(7, 4, 4, 4)	1	(8, 7, 7, 5)	1	(9, 6, 4, 3)	2	(9, 8, 5, 0)	2	(9, 9, 6, 1)	1
(7, 6, 1, 0)	1	(8, 7, 7, 7)	2	(9, 6, 5, 0)	2	(9, 8, 5, 2)	3	(9, 9, 6, 2)	3
(7, 6, 3, 0)	1	(8, 8, 0, 0)	2	(9, 6, 5, 2)	1	(9, 8, 5, 3)	1	(9, 9, 6, 3)	3
(7, 6, 5, 0)	1	(8, 8, 2, 0)	1	(9, 6, 6, 1)	2	(9, 8, 5, 4)	2	(9, 9, 6, 4)	3
(7, 6, 6, 6)	1	(8, 8, 2, 2)	1	(9, 6, 6, 3)	2	(9, 8, 6, 1)	3	(9, 9, 6, 5)	1
(7, 7, 3, 3)	1	(8, 8, 3, 2)	1	(9, 6, 6, 5)	1	(9, 8, 6, 2)	3	(9, 9, 6, 6)	2
(7, 7, 4, 3)	1	(8, 8, 4, 0)	2	(9, 6, 6, 6)	1	(9, 8, 6, 3)	4	(9, 9, 7, 1)	1
(7, 7, 5, 3)	1	(8, 8, 4, 2)	2	(9, 7, 0, 0)	1	(9, 8, 6, 4)	3	(9, 9, 7, 2)	2
(7, 7, 6, 3)	1	(8, 8, 4, 4)	2	(9, 7, 3, 1)	1	(9, 8, 6, 5)	2	(9, 9, 7, 3)	3
(7, 7, 7, 3)	1	(8, 8, 5, 2)	1	(9, 7, 3, 3)	2	(9, 8, 6, 6)	2	(9, 9, 7, 4)	3
(7, 7, 7, 7)	1	(8, 8, 5, 4)	1	(9, 7, 4, 0)	2	(9, 8, 7, 0)	1	(9, 9, 7, 5)	1
(8, 0, 0, 0)	2	(8, 8, 6, 0)	2	(9, 7, 4, 2)	3	(9, 8, 7, 1)	2	(9, 9, 7, 6)	2
(8, 2, 0, 0)	1	(8, 8, 6, 2)	2	(9, 7, 4, 3)	2	(9, 8, 7, 2)	3	(9, 9, 8, 1)	1
(8, 4, 0, 0)	1	(8, 8, 6, 4)	2	(9, 7, 4, 4)	2	(9, 8, 7, 3)	3	(9, 9, 8, 2)	1
(8, 4, 4, 0)	1	(8, 8, 6, 6)	2	(9, 7, 5, 1)	1	(9, 8, 7, 4)	3	(9, 9, 8, 3)	2
(8, 4, 4, 2)	1	(8, 8, 7, 0)	1	(9, 7, 5, 2)	1	(9, 8, 7, 5)	2	(9, 9, 8, 4)	2
(8, 4, 4, 4)	2	(8, 8, 7, 2)	2	(9, 7, 5, 3)	3	(9, 8, 7, 6)	2	(9, 9, 8, 5)	1
(8, 5, 4, 1)	1	(8, 8, 7, 4)	2	(9, 7, 5, 4)	1	(9, 8, 7, 7)	1	(9, 9, 8, 6)	2
(8, 5, 4, 3)	1	(8, 8, 7, 6)	2	(9, 7, 6, 0)	2	(9, 8, 8, 1)	2	(9, 9, 9, 3)	1
(8, 6, 0, 0)	2	(8, 8, 8, 0)	2	(9, 7, 6, 2)	3	(9, 8, 8, 2)	2	(9, 9, 9, 4)	1
(8, 6, 2, 0)	1	(8, 8, 8, 2)	2	(9, 7, 6, 3)	2	(9, 8, 8, 3)	2	(9, 9, 9, 6)	1
(8, 6, 4, 0)	2	(8, 8, 8, 4)	2	(9, 7, 6, 4)	2	(9, 8, 8, 4)	2	(9, 9, 9, 9)	1
(8, 6, 4, 2)	1	(8, 8, 8, 6)	2	(9, 7, 6, 6)	1	(9, 8, 8, 5)	2	(10, 0, 0, 0)	2
(8, 6, 4, 4)	1	(8, 8, 8, 8)	2	(9, 7, 7, 0)	1	(9, 8, 8, 6)	2	(10, 2, 0, 0)	1
(8, 6, 6, 0)	2	(9, 1, 0, 0)	1	(9, 7, 7, 2)	2	(9, 8, 8, 7)	1	(10, 4, 0, 0)	2

TABLE 4. The nonzero $d(\lambda) = \dim V_\lambda^\Gamma$ for $\lambda = (n_1, n_2, n_3, n_4)$ with $n_1 \leq 9$.

$$G = G_2(\mathbb{R}), \Gamma = G_2(\mathbb{Z}).$$

(w, v)	$m(w, v)$	(w, v)	$m(w, v)$	(w, v)	$m(w, v)$	(w, v)	$m(w, v)$	(w, v)	$m(w, v)$
(4, 2)	1	(34, 2)	2	(26, 16)	4	(30, 18)	17	(42, 12)	54
(16, 2)	1	(32, 4)	3	(24, 18)	6	(28, 20)	15	(40, 14)	60
(20, 2)	1	(30, 6)	4	(22, 20)	3	(26, 22)	3	(38, 16)	45
(12, 10)	1	(28, 8)	6	(42, 2)	5	(48, 2)	14	(36, 18)	47
(16, 8)	1	(26, 10)	3	(40, 4)	11	(46, 4)	22	(34, 20)	38
(24, 2)	1	(24, 12)	4	(38, 6)	13	(44, 6)	31	(32, 22)	24
(22, 4)	1	(22, 14)	3	(36, 8)	15	(42, 8)	31	(30, 24)	15
(20, 6)	1	(36, 2)	4	(34, 10)	16	(40, 10)	37	(28, 26)	13
(16, 10)	1	(34, 4)	4	(32, 12)	17	(38, 12)	37	(54, 2)	20
(24, 4)	1	(32, 6)	8	(30, 14)	12	(36, 14)	32	(52, 4)	39
(22, 6)	1	(30, 8)	6	(28, 16)	11	(34, 16)	28	(50, 6)	51
(20, 8)	1	(28, 10)	8	(26, 18)	9	(32, 18)	29	(48, 8)	60
(18, 10)	1	(26, 12)	6	(24, 20)	2	(30, 20)	15	(46, 10)	66
(28, 2)	3	(24, 14)	4	(44, 2)	10	(28, 22)	12	(44, 12)	72
(24, 6)	2	(22, 16)	2	(42, 4)	14	(26, 24)	5	(42, 14)	64
(22, 8)	2	(20, 18)	4	(40, 6)	18	(50, 2)	13	(40, 16)	64
(20, 10)	1	(38, 2)	3	(38, 8)	20	(48, 4)	27	(38, 18)	60
(16, 14)	2	(36, 4)	7	(36, 10)	25	(46, 6)	33	(36, 20)	45
(30, 2)	1	(34, 6)	7	(34, 12)	17	(44, 8)	41	(34, 22)	37
(28, 4)	2	(32, 8)	9	(32, 14)	20	(42, 10)	44	(32, 24)	30
(26, 6)	2	(30, 10)	9	(30, 16)	17	(40, 12)	42	(30, 26)	10
(24, 8)	2	(28, 12)	7	(28, 18)	11	(38, 14)	41	(56, 2)	29
(22, 10)	2	(26, 14)	6	(26, 20)	6	(36, 16)	41	(54, 4)	48
(20, 12)	2	(24, 16)	6	(24, 22)	6	(34, 18)	30	(52, 6)	63
(32, 2)	3	(22, 18)	2	(46, 2)	9	(32, 20)	26	(50, 8)	74
(30, 4)	3	(40, 2)	8	(44, 4)	16	(30, 22)	20	(48, 10)	88
(28, 6)	3	(38, 4)	8	(42, 6)	21	(28, 24)	6	(46, 12)	82
(26, 8)	3	(36, 6)	12	(40, 8)	28	(52, 2)	23	(44, 14)	87
(24, 10)	5	(34, 8)	13	(38, 10)	25	(50, 4)	29	(42, 16)	83
(22, 12)	2	(32, 10)	12	(36, 12)	27	(48, 6)	45	(40, 18)	72
(20, 14)	2	(30, 12)	11	(34, 14)	26	(46, 8)	52	(38, 20)	63
(18, 16)	1	(28, 14)	13	(32, 16)	19	(44, 10)	54	(36, 22)	58

TABLE 5. The nonzero $m(w, v) = \dim U_{w,v}^\Gamma$ for $v + w \leq 56$.

TABLE 6. The nonzero $S(\underline{w})$ for $\underline{w} = (w_1, w_2)$ and $w_1 \leq 43$, after Tshusima [T].

\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$
(19, 7)	1	(29, 21)	2	(35, 13)	5	(39, 15)	10	(43, 5)	3
(21, 5)	1	(29, 25)	1	(35, 15)	6	(39, 17)	8	(43, 7)	9
(21, 9)	1	(31, 3)	2	(35, 17)	5	(39, 19)	11	(43, 9)	7
(21, 13)	1	(31, 5)	1	(35, 19)	7	(39, 21)	10	(43, 11)	11
(23, 7)	1	(31, 7)	3	(35, 21)	6	(39, 23)	10	(43, 13)	11
(23, 9)	1	(31, 9)	2	(35, 23)	5	(39, 25)	10	(43, 15)	15
(23, 13)	1	(31, 11)	3	(35, 25)	5	(39, 27)	9	(43, 17)	13
(25, 5)	1	(31, 13)	4	(35, 27)	3	(39, 29)	7	(43, 19)	17
(25, 7)	1	(31, 15)	4	(35, 29)	2	(39, 31)	6	(43, 21)	14
(25, 9)	2	(31, 17)	3	(35, 31)	1	(39, 33)	4	(43, 23)	16
(25, 11)	1	(31, 19)	4	(37, 1)	1	(39, 35)	1	(43, 25)	16
(25, 13)	2	(31, 21)	3	(37, 5)	4	(39, 37)	1	(43, 27)	16
(25, 15)	1	(31, 23)	2	(37, 7)	3	(41, 1)	1	(43, 29)	14
(25, 17)	1	(31, 25)	2	(37, 9)	7	(41, 3)	1	(43, 31)	14
(25, 19)	1	(33, 5)	3	(37, 11)	5	(41, 5)	6	(43, 33)	11
(27, 3)	1	(33, 7)	2	(37, 13)	9	(41, 7)	4	(43, 35)	8
(27, 7)	2	(33, 9)	5	(37, 15)	6	(41, 9)	9	(43, 37)	7
(27, 9)	1	(33, 11)	2	(37, 17)	9	(41, 11)	6	(43, 39)	3
(27, 11)	2	(33, 13)	6	(37, 19)	8	(41, 13)	13	(45, 1)	2
(27, 13)	2	(33, 15)	4	(37, 21)	10	(41, 15)	10	(45, 3)	1
(27, 15)	2	(33, 17)	6	(37, 23)	7	(41, 17)	13	(45, 5)	8
(27, 17)	1	(33, 19)	5	(37, 25)	9	(41, 19)	11	(45, 7)	6
(27, 19)	1	(33, 21)	5	(37, 27)	6	(41, 21)	14	(45, 9)	13
(27, 21)	1	(33, 23)	3	(37, 29)	5	(41, 23)	11	(45, 11)	9
(29, 5)	2	(33, 25)	4	(37, 31)	4	(41, 25)	15	(45, 13)	17
(29, 7)	1	(33, 27)	2	(37, 33)	2	(41, 27)	11	(45, 15)	13
(29, 9)	3	(33, 29)	1	(39, 3)	3	(41, 29)	11	(45, 17)	19
(29, 11)	1	(35, 3)	2	(39, 5)	2	(41, 31)	9	(45, 19)	17
(29, 13)	4	(35, 5)	1	(39, 7)	7	(41, 33)	8	(45, 21)	21
(29, 15)	2	(35, 7)	5	(39, 9)	5	(41, 35)	4	(45, 23)	16
(29, 17)	3	(35, 9)	4	(39, 11)	8	(41, 37)	3	(45, 25)	22
(29, 19)	2	(35, 11)	5	(39, 13)	8	(43, 3)	5	(45, 27)	18

TABLE 7. The nonzero $S(\underline{w})$ for $\underline{w} = (w_1, w_2, w_3)$ and $w_1 \leq 29$.

\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$
(23, 13, 5)	1	(27, 17, 11)	1	(29, 11, 5)	1	(29, 21, 19)	1
(23, 15, 3)	1	(27, 17, 13)	1	(29, 13, 3)	1	(29, 23, 1)	1
(23, 15, 7)	1	(27, 19, 3)	2	(29, 13, 5)	1	(29, 23, 3)	2
(23, 17, 5)	1	(27, 19, 5)	2	(29, 13, 7)	3	(29, 23, 5)	5
(23, 17, 9)	1	(27, 19, 7)	3	(29, 13, 9)	1	(29, 23, 7)	5
(23, 19, 3)	1	(27, 19, 9)	3	(29, 15, 1)	1	(29, 23, 9)	6
(23, 19, 11)	1	(27, 19, 11)	3	(29, 15, 5)	3	(29, 23, 11)	7
(25, 13, 3)	1	(27, 19, 13)	2	(29, 15, 7)	2	(29, 23, 13)	5
(25, 13, 7)	1	(27, 19, 15)	1	(29, 15, 9)	3	(29, 23, 15)	5
(25, 15, 5)	1	(27, 21, 1)	1	(29, 15, 13)	1	(29, 23, 17)	3
(25, 15, 9)	1	(27, 21, 5)	4	(29, 17, 3)	3	(29, 23, 19)	1
(25, 17, 3)	2	(27, 21, 7)	2	(29, 17, 5)	1	(29, 25, 3)	3
(25, 17, 7)	2	(27, 21, 9)	4	(29, 17, 7)	6	(29, 25, 5)	3
(25, 17, 11)	1	(27, 21, 11)	2	(29, 17, 9)	3	(29, 25, 7)	7
(25, 19, 1)	1	(27, 21, 13)	3	(29, 17, 11)	3	(29, 25, 9)	4
(25, 19, 5)	2	(27, 21, 15)	1	(29, 17, 13)	1	(29, 25, 11)	7
(25, 19, 9)	2	(27, 21, 17)	1	(29, 19, 1)	1	(29, 25, 13)	4
(25, 19, 13)	1	(27, 23, 3)	1	(29, 19, 3)	1	(29, 25, 15)	5
(25, 21, 3)	2	(27, 23, 5)	3	(29, 19, 5)	6	(29, 25, 17)	3
(25, 21, 7)	2	(27, 23, 7)	1	(29, 19, 7)	3	(29, 25, 19)	2
(25, 21, 11)	2	(27, 23, 9)	2	(29, 19, 9)	7	(29, 25, 21)	1
(25, 21, 15)	1	(27, 23, 11)	2	(29, 19, 11)	4	(29, 27, 1)	1
(27, 9, 5)	1	(27, 23, 13)	1	(29, 19, 13)	5	(29, 27, 5)	1
(27, 13, 5)	2	(27, 23, 15)	1	(29, 19, 15)	1	(29, 27, 7)	2
(27, 13, 7)	1	(27, 23, 17)	1	(29, 19, 17)	1	(29, 27, 9)	3
(27, 13, 9)	1	(27, 25, 5)	2	(29, 21, 3)	5	(29, 27, 11)	1
(27, 15, 3)	1	(27, 25, 7)	1	(29, 21, 5)	1	(29, 27, 13)	2
(27, 15, 5)	1	(27, 25, 9)	1	(29, 21, 7)	10	(29, 27, 15)	1
(27, 15, 7)	2	(27, 25, 11)	1	(29, 21, 9)	4	(29, 27, 17)	1
(27, 15, 9)	1	(27, 25, 13)	1	(29, 21, 11)	8	(29, 27, 19)	1
(27, 17, 5)	4	(27, 25, 15)	1	(29, 21, 13)	4	(31, 9, 5)	1
(27, 17, 7)	1	(27, 25, 17)	1	(29, 21, 15)	5	(31, 11, 3)	1
(27, 17, 9)	3	(29, 9, 7)	1	(29, 21, 17)	1	(31, 11, 7)	1

TABLE 8. The nonzero $S(\underline{w})$ for $\underline{w} = (w_1, w_2, w_3, w_4)$ and $w_1 \leq 27$.

\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$	\underline{w}	$S(\underline{w})$
(25, 17, 9, 5)	1	(27, 17, 13, 7)	2	(27, 21, 19, 7)	1	(27, 23, 21, 9)	1
(25, 17, 13, 5)	1	(27, 19, 9, 5)	1	(27, 21, 19, 9)	1	(27, 25, 9, 3)	2
(25, 19, 9, 3)	1	(27, 19, 11, 3)	2	(27, 21, 19, 11)	1	(27, 25, 11, 1)	1
(25, 19, 11, 5)	1	(27, 19, 11, 5)	1	(27, 23, 7, 3)	2	(27, 25, 11, 3)	1
(25, 19, 13, 3)	1	(27, 19, 13, 1)	1	(27, 23, 9, 1)	1	(27, 25, 11, 5)	2
(25, 19, 13, 5)	1	(27, 19, 13, 3)	1	(27, 23, 9, 5)	2	(27, 25, 13, 3)	5
(25, 19, 13, 7)	1	(27, 19, 13, 5)	4	(27, 23, 11, 3)	5	(27, 25, 13, 5)	1
(25, 19, 13, 9)	1	(27, 19, 13, 7)	1	(27, 23, 11, 5)	1	(27, 25, 13, 7)	4
(25, 19, 15, 5)	1	(27, 19, 13, 9)	3	(27, 23, 11, 7)	4	(27, 25, 13, 9)	1
(25, 21, 11, 7)	1	(27, 19, 15, 3)	2	(27, 23, 13, 1)	4	(27, 25, 15, 1)	3
(25, 21, 13, 5)	1	(27, 19, 15, 5)	1	(27, 23, 13, 3)	1	(27, 25, 15, 3)	2
(25, 21, 13, 7)	1	(27, 19, 15, 7)	1	(27, 23, 13, 5)	6	(27, 25, 15, 5)	5
(25, 21, 15, 3)	1	(27, 19, 15, 9)	1	(27, 23, 13, 7)	3	(27, 25, 15, 7)	3
(25, 21, 15, 5)	1	(27, 19, 17, 5)	1	(27, 23, 13, 9)	6	(27, 25, 15, 9)	5
(25, 21, 15, 7)	2	(27, 19, 17, 9)	1	(27, 23, 15, 3)	7	(27, 25, 15, 11)	1
(25, 21, 15, 9)	1	(27, 21, 9, 3)	2	(27, 23, 15, 5)	3	(27, 25, 17, 3)	7
(25, 21, 17, 5)	1	(27, 21, 9, 7)	1	(27, 23, 15, 7)	7	(27, 25, 17, 5)	2
(25, 21, 17, 7)	1	(27, 21, 11, 3)	1	(27, 23, 15, 9)	4	(27, 25, 17, 7)	7
(25, 21, 17, 9)	1	(27, 21, 11, 5)	2	(27, 23, 15, 11)	5	(27, 25, 17, 9)	4
(25, 23, 9, 3)	1	(27, 21, 11, 7)	2	(27, 23, 15, 13)	1	(27, 25, 17, 11)	5
(25, 23, 11, 1)	1	(27, 21, 13, 3)	5	(27, 23, 17, 1)	5	(27, 25, 17, 13)	1
(25, 23, 11, 5)	2	(27, 21, 13, 5)	2	(27, 23, 17, 3)	2	(27, 25, 19, 1)	3
(25, 23, 13, 3)	1	(27, 21, 13, 7)	6	(27, 23, 17, 5)	6	(27, 25, 19, 3)	2
(25, 23, 13, 7)	1	(27, 21, 13, 9)	2	(27, 23, 17, 7)	5	(27, 25, 19, 5)	5
(25, 23, 15, 1)	1	(27, 21, 15, 1)	1	(27, 23, 17, 9)	7	(27, 25, 19, 7)	3
(25, 23, 15, 5)	3	(27, 21, 15, 3)	2	(27, 23, 17, 11)	3	(27, 25, 19, 9)	6
(25, 23, 15, 9)	1	(27, 21, 15, 5)	4	(27, 23, 17, 13)	4	(27, 25, 19, 11)	3
(25, 23, 15, 11)	1	(27, 21, 15, 7)	4	(27, 23, 19, 3)	5	(27, 25, 19, 13)	3
(25, 23, 17, 3)	1	(27, 21, 15, 9)	4	(27, 23, 19, 5)	1	(27, 25, 21, 3)	4
(25, 23, 17, 5)	1	(27, 21, 15, 11)	2	(27, 23, 19, 7)	6	(27, 25, 21, 7)	4
(25, 23, 17, 7)	1	(27, 21, 17, 3)	5	(27, 23, 19, 9)	2	(27, 25, 21, 9)	2
(25, 23, 17, 11)	1	(27, 21, 17, 7)	6	(27, 23, 19, 11)	3	(27, 25, 21, 11)	3
(25, 23, 19, 5)	1	(27, 21, 17, 9)	2	(27, 23, 19, 13)	1	(27, 25, 21, 13)	1
(27, 17, 9, 3)	1	(27, 21, 17, 11)	3	(27, 23, 19, 15)	1	(27, 25, 21, 15)	1
(27, 17, 9, 7)	1	(27, 21, 19, 3)	1	(27, 23, 21, 1)	1	(27, 25, 23, 3)	1
(27, 17, 13, 3)	2	(27, 21, 19, 5)	1	(27, 23, 21, 5)	1	(27, 25, 23, 9)	1

TABLE 9. The nonzero $O(\underline{w})$ for $\underline{w} = (w_1, w_2, w_3, w_4)$ and $0 < w_4 < w_1 \leq 30$.

\underline{w}	$O(\underline{w})$	\underline{w}	$O(\underline{w})$	\underline{w}	$O(\underline{w})$	\underline{w}	$O(\underline{w})$
(24, 18, 10, 4)	1	(28, 24, 14, 2)	2	(30, 22, 14, 2)	2	(30, 26, 16, 12)	1
(24, 20, 14, 2)	1	(28, 24, 14, 10)	1	(30, 22, 14, 6)	3	(30, 26, 18, 2)	3
(26, 18, 10, 2)	1	(28, 24, 16, 4)	1	(30, 22, 16, 4)	2	(30, 26, 18, 6)	2
(26, 18, 14, 6)	1	(28, 24, 16, 12)	1	(30, 22, 16, 8)	1	(30, 26, 18, 10)	1
(26, 20, 10, 4)	1	(28, 24, 18, 2)	1	(30, 22, 18, 2)	1	(30, 26, 18, 14)	1
(26, 20, 14, 8)	1	(28, 24, 18, 6)	1	(30, 22, 18, 6)	1	(30, 26, 20, 4)	3
(26, 22, 10, 6)	1	(28, 24, 20, 4)	1	(30, 22, 18, 10)	1	(30, 26, 20, 8)	1
(26, 22, 14, 2)	1	(28, 24, 20, 8)	1	(30, 24, 8, 2)	1	(30, 26, 22, 2)	1
(26, 24, 14, 4)	1	(28, 26, 12, 2)	1	(30, 24, 10, 4)	3	(30, 26, 22, 6)	2
(26, 24, 16, 2)	1	(28, 26, 14, 4)	1	(30, 24, 12, 2)	2	(30, 26, 22, 10)	1
(26, 24, 18, 8)	1	(28, 26, 16, 2)	2	(30, 24, 12, 6)	2	(30, 28, 10, 4)	1
(26, 24, 20, 6)	1	(28, 26, 18, 8)	1	(30, 24, 14, 4)	2	(30, 28, 10, 8)	1
(28, 16, 10, 6)	1	(28, 26, 20, 6)	1	(30, 24, 14, 8)	3	(30, 28, 12, 2)	1
(28, 18, 8, 2)	1	(28, 26, 22, 4)	1	(30, 24, 16, 2)	3	(30, 28, 14, 4)	3
(28, 18, 12, 2)	1	(30, 14, 8, 4)	1	(30, 24, 16, 6)	2	(30, 28, 14, 12)	1
(28, 18, 14, 4)	1	(30, 16, 10, 4)	1	(30, 24, 16, 10)	2	(30, 28, 16, 2)	2
(28, 20, 10, 2)	1	(30, 18, 8, 4)	1	(30, 24, 18, 4)	4	(30, 28, 16, 6)	1
(28, 20, 12, 4)	1	(30, 18, 10, 2)	1	(30, 24, 18, 8)	1	(30, 28, 18, 4)	1
(28, 20, 14, 2)	1	(30, 18, 10, 6)	1	(30, 24, 18, 12)	2	(30, 28, 18, 8)	2
(28, 20, 14, 6)	1	(30, 18, 12, 4)	1	(30, 24, 20, 2)	2	(30, 28, 20, 2)	1
(28, 20, 16, 4)	1	(30, 18, 14, 2)	1	(30, 24, 20, 6)	2	(30, 28, 20, 6)	3
(28, 20, 16, 8)	1	(30, 18, 14, 6)	1	(30, 24, 20, 10)	1	(30, 28, 20, 10)	1
(28, 22, 8, 2)	1	(30, 20, 6, 4)	1	(30, 24, 20, 14)	1	(30, 28, 22, 4)	4
(28, 22, 10, 4)	1	(30, 20, 10, 4)	1	(30, 24, 22, 8)	1	(30, 28, 22, 8)	1
(28, 22, 12, 2)	1	(30, 20, 10, 8)	1	(30, 24, 22, 16)	1	(30, 28, 22, 12)	1
(28, 22, 12, 6)	1	(30, 20, 12, 2)	1	(30, 26, 6, 2)	1	(32, 16, 8, 4)	1
(28, 22, 14, 8)	1	(30, 20, 14, 4)	3	(30, 26, 8, 4)	1	(32, 16, 10, 2)	1
(28, 22, 16, 2)	1	(30, 20, 14, 8)	1	(30, 26, 10, 2)	1	(32, 16, 10, 6)	1
(28, 22, 16, 6)	1	(30, 20, 14, 12)	1	(30, 26, 10, 6)	2	(32, 18, 8, 2)	1
(28, 22, 16, 10)	1	(30, 20, 16, 2)	1	(30, 26, 12, 4)	2	(32, 18, 8, 6)	1
(28, 22, 18, 4)	1	(30, 20, 16, 6)	1	(30, 26, 12, 8)	1	(32, 18, 10, 4)	2
(28, 24, 8, 4)	1	(30, 20, 18, 8)	1	(30, 26, 14, 2)	3	(32, 18, 10, 8)	1
(28, 24, 10, 2)	1	(30, 22, 8, 4)	1	(30, 26, 14, 6)	1	(32, 18, 12, 2)	1
(28, 24, 10, 6)	1	(30, 22, 10, 2)	2	(30, 26, 14, 10)	2	(32, 18, 12, 6)	1
(28, 24, 12, 4)	1	(30, 22, 10, 6)	1	(30, 26, 16, 4)	2	(32, 18, 14, 4)	1
(28, 24, 12, 8)	1	(30, 22, 12, 4)	1	(30, 26, 16, 8)	1	(32, 18, 14, 8)	1

TABLE 10. The nonzero $O(\underline{w}) = 2 \cdot O(w_1, w_2, w_3, 0) + O^*(w_1, w_2, w_3)$ for $w_1 \leq 34$.

\underline{w}	$O(\underline{w})$	\underline{w}	$O(\underline{w})$	\underline{w}	$O(\underline{w})$	\underline{w}	$O(\underline{w})$
(24, 16, 8, 0)	1	(30, 28, 10, 0)	2	(32, 30, 10, 0)	2	(34, 28, 26, 0)	2
(26, 16, 10, 0)	1	(30, 28, 14, 0)	3	(32, 30, 14, 0)	4	(34, 30, 4, 0)	2
(26, 20, 6, 0)	1	(30, 28, 18, 0)	5	(32, 30, 18, 0)	6	(34, 30, 8, 0)	2
(26, 20, 10, 0)	1	(30, 28, 26, 0)	1	(32, 30, 26, 0)	6	(34, 30, 12, 0)	7
(26, 20, 14, 0)	1	(32, 12, 8, 0)	1	(34, 12, 6, 0)	1	(34, 30, 16, 0)	14
(26, 24, 10, 0)	1	(32, 14, 10, 0)	1	(34, 14, 8, 0)	1	(34, 30, 20, 0)	6
(26, 24, 14, 0)	1	(32, 16, 4, 0)	1	(34, 16, 6, 0)	1	(34, 30, 24, 0)	7
(26, 24, 18, 0)	1	(32, 16, 8, 0)	1	(34, 16, 10, 0)	3	(34, 32, 2, 0)	1
(28, 14, 6, 0)	1	(32, 16, 12, 0)	1	(34, 16, 14, 0)	1	(34, 32, 6, 0)	2
(28, 16, 8, 0)	1	(32, 18, 6, 0)	1	(34, 18, 4, 0)	1	(34, 32, 10, 0)	6
(28, 18, 10, 0)	1	(32, 18, 10, 0)	1	(34, 18, 8, 0)	1	(34, 32, 14, 0)	8
(28, 20, 8, 0)	1	(32, 18, 14, 0)	3	(34, 18, 12, 0)	3	(34, 32, 18, 0)	13
(28, 20, 12, 0)	1	(32, 20, 4, 0)	1	(34, 20, 6, 0)	3	(34, 32, 22, 0)	3
(28, 22, 14, 0)	2	(32, 20, 8, 0)	2	(34, 20, 10, 0)	3	(34, 32, 26, 0)	14
(28, 24, 4, 0)	1	(32, 20, 12, 0)	2	(34, 20, 14, 0)	8	(36, 12, 8, 0)	1
(28, 24, 12, 0)	1	(32, 20, 16, 0)	3	(34, 20, 18, 0)	2	(36, 14, 6, 0)	1
(28, 24, 16, 0)	3	(32, 22, 6, 0)	1	(34, 22, 4, 0)	1	(36, 14, 10, 0)	1
(28, 26, 18, 0)	2	(32, 22, 10, 0)	4	(34, 22, 8, 0)	3	(36, 16, 4, 0)	1
(30, 16, 6, 0)	1	(32, 22, 14, 0)	1	(34, 22, 12, 0)	3	(36, 16, 8, 0)	3
(30, 16, 10, 0)	1	(32, 22, 18, 0)	3	(34, 22, 16, 0)	5	(36, 16, 12, 0)	2
(30, 16, 14, 0)	1	(32, 24, 4, 0)	1	(34, 24, 6, 0)	3	(36, 18, 6, 0)	2
(30, 18, 8, 0)	1	(32, 24, 8, 0)	5	(34, 24, 10, 0)	11	(36, 18, 10, 0)	5
(30, 20, 6, 0)	1	(32, 24, 12, 0)	5	(34, 24, 14, 0)	7	(36, 18, 14, 0)	4
(30, 20, 10, 0)	4	(32, 24, 16, 0)	4	(34, 24, 18, 0)	12	(36, 20, 4, 0)	2
(30, 20, 14, 0)	1	(32, 24, 20, 0)	6	(34, 24, 22, 0)	2	(36, 20, 8, 0)	4
(30, 20, 18, 0)	1	(32, 26, 6, 0)	2	(34, 26, 4, 0)	1	(36, 20, 12, 0)	6
(30, 22, 8, 0)	1	(32, 26, 10, 0)	4	(34, 26, 8, 0)	6	(36, 20, 16, 0)	6
(30, 22, 12, 0)	1	(32, 26, 14, 0)	8	(34, 26, 12, 0)	9	(36, 22, 6, 0)	3
(30, 24, 6, 0)	2	(32, 26, 18, 0)	3	(34, 26, 16, 0)	7	(36, 22, 10, 0)	6
(30, 24, 10, 0)	2	(32, 26, 22, 0)	5	(34, 26, 20, 0)	12	(36, 22, 14, 0)	10
(30, 24, 14, 0)	5	(32, 28, 4, 0)	2	(34, 26, 24, 0)	2	(36, 22, 18, 0)	6
(30, 24, 18, 0)	2	(32, 28, 8, 0)	2	(34, 28, 6, 0)	6	(36, 24, 4, 0)	3
(30, 26, 8, 0)	1	(32, 28, 12, 0)	5	(34, 28, 10, 0)	8	(36, 24, 8, 0)	8
(30, 26, 12, 0)	1	(32, 28, 16, 0)	9	(34, 28, 14, 0)	16	(36, 24, 12, 0)	13
(30, 26, 16, 0)	3	(32, 28, 20, 0)	4	(34, 28, 18, 0)	11	(36, 24, 16, 0)	16
(30, 28, 2, 0)	1	(32, 28, 24, 0)	3	(34, 28, 22, 0)	14	(36, 24, 20, 0)	12

TABLE 11. The nonzero $G_2(\underline{w})$ for $\underline{w} = (w, v)$ and $w + v \leq 58$.

(w, v)	$G_2(\underline{w})$	(w, v)	$G_2(\underline{w})$	(w, v)	$G_2(\underline{w})$	(w, v)	$G_2(\underline{w})$	(w, v)	$G_2(\underline{w})$
(16, 8)	1	(30, 8)	4	(44, 2)	7	(28, 22)	12	(44, 12)	72
(20, 6)	1	(28, 10)	8	(42, 4)	13	(26, 24)	4	(42, 14)	61
(16, 10)	1	(26, 12)	6	(40, 6)	18	(50, 2)	11	(40, 16)	64
(24, 4)	1	(24, 14)	4	(38, 8)	18	(48, 4)	27	(38, 18)	58
(20, 8)	1	(20, 18)	3	(36, 10)	25	(46, 6)	29	(36, 20)	45
(18, 10)	1	(38, 2)	2	(34, 12)	15	(44, 8)	41	(34, 22)	34
(28, 2)	1	(36, 4)	7	(32, 14)	20	(42, 10)	42	(32, 24)	30
(24, 6)	2	(34, 6)	5	(30, 16)	15	(40, 12)	42	(30, 26)	7
(22, 8)	1	(32, 8)	9	(28, 18)	11	(38, 14)	39	(56, 2)	25
(20, 10)	1	(30, 10)	8	(26, 20)	6	(36, 16)	41	(54, 4)	44
(16, 14)	1	(28, 12)	7	(24, 22)	4	(34, 18)	27	(52, 6)	63
(28, 4)	2	(26, 14)	6	(46, 2)	7	(32, 20)	26	(50, 8)	72
(26, 6)	2	(24, 16)	6	(44, 4)	16	(30, 22)	18	(48, 10)	88
(24, 8)	2	(40, 2)	5	(42, 6)	19	(28, 24)	6	(46, 12)	76
(22, 10)	1	(38, 4)	6	(40, 8)	28	(52, 2)	19	(44, 14)	87
(20, 12)	2	(36, 6)	12	(38, 10)	23	(50, 4)	27	(42, 16)	81
(32, 2)	1	(34, 8)	12	(36, 12)	27	(48, 6)	45	(40, 18)	72
(30, 4)	2	(32, 10)	12	(34, 14)	24	(46, 8)	48	(38, 20)	60
(28, 6)	3	(30, 12)	9	(32, 16)	19	(44, 10)	54	(36, 22)	58
(26, 8)	3	(28, 14)	13	(30, 18)	15	(42, 12)	52	(34, 24)	29
(24, 10)	5	(26, 16)	4	(28, 20)	15	(40, 14)	60	(32, 26)	26
(20, 14)	2	(24, 18)	6	(26, 22)	3	(38, 16)	42	(30, 28)	6
(34, 2)	1	(42, 2)	3	(48, 2)	11	(36, 18)	47	(58, 2)	25
(32, 4)	3	(40, 4)	11	(46, 4)	18	(34, 20)	36	(56, 4)	54
(30, 6)	3	(38, 6)	12	(44, 6)	31	(32, 22)	24	(54, 6)	69
(28, 8)	6	(36, 8)	15	(42, 8)	29	(30, 24)	12	(52, 8)	93
(26, 10)	3	(34, 10)	14	(40, 10)	37	(28, 26)	11	(50, 10)	92
(24, 12)	4	(32, 12)	17	(38, 12)	35	(54, 2)	16	(48, 12)	104
(22, 14)	2	(30, 14)	10	(36, 14)	32	(52, 4)	39	(46, 14)	102
(36, 2)	2	(28, 16)	11	(34, 16)	26	(50, 6)	49	(44, 16)	96
(34, 4)	3	(26, 18)	9	(32, 18)	29	(48, 8)	60	(42, 18)	89
(32, 6)	8	(24, 20)	2	(30, 20)	12	(46, 10)	62	(40, 20)	88

TABLE 12. The nonempty $\Pi_{w_1, w_2, w_3}(\mathrm{SO}(7))$ for $w_1 \leq 23$

(w_1, w_2, w_3)	$\Pi_{w_1, w_2, w_3}(\mathrm{SO}(7))$	(w_1, w_2, w_3)	$\Pi_{w_1, w_2, w_3}(\mathrm{SO}(7))$
(5,3,1)	[6]	(21,19,17)	$\Delta_{19}[3]$
(13,11,9)	$\Delta_{11}[3]$	(23,9,1)	$\Delta_{23,9} \oplus [2]$
(17,3,1)	$\Delta_{17} \oplus [4]$	(23,11,7)	$\Delta_{23,7} \oplus \Delta_{11}$
(17,11,1)	$\Delta_{17} \oplus \Delta_{11} \oplus [2]$	(23,11,9)	$\Delta_{23,9} \oplus \Delta_{11}$
(17,15,1)	$\Delta_{17} \oplus \Delta_{15} \oplus [2]$	(23,13,1)	$\Delta_{23,13} \oplus [2]$
(17,15,13)	$\Delta_{15}[3]$	(23,13,5)	$\Delta_{23,13,5}$
(19,11,7)	$\Delta_{19,7} \oplus \Delta_{11}$	(23,15,3)	$\Delta_{23,15,3}$
(19,15,7)	$\Delta_{19,7} \oplus \Delta_{15}$	(23,15,7)	$\Delta_{23,7} \oplus \Delta_{15}, \Delta_{23,15,7}$
(19,17,7)	$\Delta_{19,7} \oplus \Delta_{17}$	(23,15,9)	$\Delta_{23,9} \oplus \Delta_{15}$
(19,17,15)	$\Delta_{17}[3]$	(23,15,13)	$\Delta_{23,13} \oplus \Delta_{15}$
(21,3,1)	$\Delta_{21} \oplus [4]$	(23,17,5)	$\Delta_{23,17,5}$
(21,11,1)	$\Delta_{21} \oplus \Delta_{11} \oplus [2]$	(23,17,7)	$\Delta_{23,7} \oplus \Delta_{17}$
(21,11,5)	$\Delta_{21,5} \oplus \Delta_{11}$	(23,17,9)	$\Delta_{23,9} \oplus \Delta_{17}, \Delta_{23,17,9}$
(21,11,9)	$\Delta_{21,9} \oplus \Delta_{11}$	(23,17,13)	$\Delta_{23,13} \oplus \Delta_{17}$
(21,15,1)	$\Delta_{21} \oplus \Delta_{15} \oplus [2]$	(23,19,3)	$\Delta_{23,19,3}$
(21,15,5)	$\Delta_{21,5} \oplus \Delta_{15}$	(23,19,7)	$\Delta_{23,7} \oplus \Delta_{19}$
(21,15,9)	$\Delta_{21,9} \oplus \Delta_{15}$	(23,19,9)	$\Delta_{23,9} \oplus \Delta_{19}$
(21,15,13)	$\Delta_{21,13} \oplus \Delta_{15}$	(23,19,11)	$\Delta_{23,19,11}$
(21,17,5)	$\Delta_{21,5} \oplus \Delta_{17}$	(23,19,13)	$\Delta_{23,13} \oplus \Delta_{19}$
(21,17,9)	$\Delta_{21,9} \oplus \Delta_{17}$	(23,21,1)	$\mathrm{Sym}^2 \Delta_{11}[2]$
(21,17,13)	$\Delta_{21,13} \oplus \Delta_{17}$	(23,21,7)	$\Delta_{23,7} \oplus \Delta_{21}$
(21,19,1)	$\Delta_{21} \oplus \Delta_{19} \oplus [2]$	(23,21,9)	$\Delta_{23,9} \oplus \Delta_{21}$
(21,19,5)	$\Delta_{21,5} \oplus \Delta_{19}$	(23,21,13)	$\Delta_{23,13} \oplus \Delta_{21}$
(21,19,9)	$\Delta_{21,9} \oplus \Delta_{19}$	(23,21,19)	$\Delta_{21}[3]$
(21,19,13)	$\Delta_{21,13} \oplus \Delta_{19}$		

TABLE 13. The nonempty $\Pi_{25,w_2,w_3}(\mathrm{SO}(7))$

(w_1, w_2, w_3)	$\Pi_{w_1,w_2,w_3}(\mathrm{SO}(7))$	(w_1, w_2, w_3)	$\Pi_{w_1,w_2,w_3}(\mathrm{SO}(7))$
(25,3,1)	$\Delta_{25} \oplus [4]$	(25,19,9)	$\Delta_{25,9}^2 \oplus \Delta_{19}, \Delta_{25,19,9}^2$
(25,7,1)	$\Delta_{25,7} \oplus [2]$	(25,19,11)	$\Delta_{25,11} \oplus \Delta_{19}$
(25,11,1)	$\Delta_{25,11} \oplus [2], \Delta_{25} \oplus \Delta_{11} \oplus [2]$	(25,19,13)	$\Delta_{25,13}^2 \oplus \Delta_{19}, \Delta_{25,19,13}$
(25,11,5)	$\Delta_{25,5} \oplus \Delta_{11}$	(25,19,15)	$\Delta_{25,15} \oplus \Delta_{19}$
(25,11,7)	$\Delta_{25,7} \oplus \Delta_{11}$	(25,19,17)	$\Delta_{25,17} \oplus \Delta_{19}$
(25,11,9)	$\Delta_{25,9}^2 \oplus \Delta_{11}$	(25,21,3)	$\Delta_{25,21,3}^2$
(25,13,3)	$\Delta_{25,13,3}$	(25,21,5)	$\Delta_{25,5} \oplus \Delta_{21}$
(25,13,7)	$\Delta_{25,13,7}$	(25,21,7)	$\Delta_{25,7} \oplus \Delta_{21}, \Delta_{25,21,7}^2$
(25,15,1)	$\Delta_{25,15} \oplus [2], \Delta_{25} \oplus \Delta_{15} \oplus [2]$	(25,21,9)	$\Delta_{25,9}^2 \oplus \Delta_{21}$
(25,15,5)	$\Delta_{25,5} \oplus \Delta_{15}, \Delta_{25,15,5}$	(25,21,11)	$\Delta_{25,11} \oplus \Delta_{21}, \Delta_{25,21,11}^2$
(25,15,7)	$\Delta_{25,7} \oplus \Delta_{15}$	(25,21,13)	$\Delta_{25,13}^2 \oplus \Delta_{21}$
(25,15,9)	$\Delta_{25,9}^2 \oplus \Delta_{15}, \Delta_{25,15,9}$	(25,21,15)	$\Delta_{25,15} \oplus \Delta_{21}, \Delta_{25,21,15}$
(25,15,11)	$\Delta_{25,11} \oplus \Delta_{15}$	(25,21,17)	$\Delta_{25,17} \oplus \Delta_{21}$
(25,15,13)	$\Delta_{25,13}^2 \oplus \Delta_{15}$	(25,21,19)	$\Delta_{25,19} \oplus \Delta_{21}$
(25,17,3)	$\Delta_{25,17,3}^2$	(25,23,1)	$\Delta_{25} \oplus \Delta_{23}^2 \oplus [2]$
(25,17,5)	$\Delta_{25,5} \oplus \Delta_{17}$	(25,23,5)	$\Delta_{25,5} \oplus \Delta_{23}^2$
(25,17,7)	$\Delta_{25,7} \oplus \Delta_{17}, \Delta_{25,17,7}^2$	(25,23,7)	$\Delta_{25,7} \oplus \Delta_{23}^2$
(25,17,9)	$\Delta_{25,9}^2 \oplus \Delta_{17}$	(25,23,9)	$\Delta_{25,9}^2 \oplus \Delta_{23}^2$
(25,17,11)	$\Delta_{25,11} \oplus \Delta_{17}, \Delta_{25,17,11}$	(25,23,11)	$\Delta_{25,11} \oplus \Delta_{23}^2$
(25,17,13)	$\Delta_{25,13}^2 \oplus \Delta_{17}$	(25,23,13)	$\Delta_{25,13}^2 \oplus \Delta_{23}^2$
(25,17,15)	$\Delta_{25,15} \oplus \Delta_{17}$	(25,23,15)	$\Delta_{25,15} \oplus \Delta_{23}^2$
(25,19,1)	$\Delta_{25,19} \oplus [2], \Delta_{25} \oplus \Delta_{19} \oplus [2], \Delta_{25,19,1}$	(25,23,17)	$\Delta_{25,17} \oplus \Delta_{23}^2$
(25,19,5)	$\Delta_{25,5} \oplus \Delta_{19}, \Delta_{25,19,5}^2$	(25,23,19)	$\Delta_{25,19} \oplus \Delta_{23}^2$
(25,19,7)	$\Delta_{25,7} \oplus \Delta_{19}$	(25,23,21)	$\Delta_{23}^2[3]$

TABLE 14. The nonempty $\Pi_{w_1, w_2, w_3, w_4}(\mathrm{SO}(9))$ for $w_1 \leq 23$

(w_1, w_2, w_3, w_4)	$\Pi_{w_1, w_2, w_3, w_4}(\mathrm{SO}(9))$	(w_1, w_2, w_3, w_4)	$\Pi_{w_1, w_2, w_3, w_4}(\mathrm{SO}(9))$
(7, 5, 3, 1)	[8]	(23, 17, 15, 5)	$\Delta_{23,17,5} \oplus \Delta_{15}$
(11, 5, 3, 1)	$\Delta_{11} \oplus [6]$	(23, 17, 15, 9)	$\Delta_{23,17,9} \oplus \Delta_{15}$
(15, 5, 3, 1)	$\Delta_{15} \oplus [6]$	(23, 17, 15, 13)	$\Delta_{23}^2 \oplus \Delta_{15}[3]$
(15, 13, 11, 9)	$\Delta_{15} \oplus \Delta_{11}[3]$	(23, 19, 9, 7)	$\Delta_{23,9} \oplus \Delta_{19,7}$
(17, 13, 11, 9)	$\Delta_{17} \oplus \Delta_{11}[3]$	(23, 19, 11, 3)	$\Delta_{23,19,3} \oplus \Delta_{11}$
(19, 5, 3, 1)	$\Delta_{19} \oplus [6]$	(23, 19, 11, 7)	$\Delta_{23}^2 \oplus \Delta_{19,7} \oplus \Delta_{11}$
(19, 13, 11, 9)	$\Delta_{19} \oplus \Delta_{11}[3]$	(23, 19, 13, 7)	$\Delta_{23,13} \oplus \Delta_{19,7}$
(19, 17, 3, 1)	$\Delta_{19} \oplus \Delta_{17} \oplus [4]$	(23, 19, 15, 3)	$\Delta_{23,19,3} \oplus \Delta_{15}$
(19, 17, 7, 1)	$\Delta_{19,7} \oplus \Delta_{17} \oplus [2]$	(23, 19, 15, 7)	$\Delta_{23}^2 \oplus \Delta_{19,7} \oplus \Delta_{15}$
(19, 17, 11, 1)	$\Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11} \oplus [2]$	(23, 19, 15, 11)	$\Delta_{23,19,11} \oplus \Delta_{15}$
(19, 17, 15, 1)	$\Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [2]$	(23, 19, 17, 3)	$\Delta_{23,19,3} \oplus \Delta_{17}$
(19, 17, 15, 13)	$\Delta_{19} \oplus \Delta_{15}[3]$	(23, 19, 17, 7)	$\Delta_{23}^2 \oplus \Delta_{19,7} \oplus \Delta_{17}$
(21, 13, 11, 9)	$\Delta_{21} \oplus \Delta_{11}[3]$	(23, 19, 17, 11)	$\Delta_{23,19,11} \oplus \Delta_{17}$
(21, 17, 5, 1)	$\Delta_{21,5} \oplus \Delta_{17} \oplus [2]$	(23, 19, 17, 15)	$\Delta_{23}^2 \oplus \Delta_{17}[3]$
(21, 17, 9, 1)	$\Delta_{21,9} \oplus \Delta_{17} \oplus [2]$	(23, 21, 3, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus [4]$
(21, 17, 13, 1)	$\Delta_{21,13} \oplus \Delta_{17} \oplus [2]$	(23, 21, 7, 1)	$\Delta_{23,7} \oplus \Delta_{21} \oplus [2]$
(21, 17, 15, 13)	$\Delta_{21} \oplus \Delta_{15}[3]$	(23, 21, 7, 5)	$\Delta_{23,7} \oplus \Delta_{21,5}$
(21, 19, 9, 7)	$\Delta_{21,9} \oplus \Delta_{19,7}$	(23, 21, 9, 5)	$\Delta_{23,9} \oplus \Delta_{21,5}$
(21, 19, 11, 7)	$\Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{11}$	(23, 21, 11, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{11} \oplus [2]$
(21, 19, 13, 7)	$\Delta_{21,13} \oplus \Delta_{19,7}$	(23, 21, 11, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{11}$
(21, 19, 15, 7)	$\Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{15}$	(23, 21, 11, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{11}$
(21, 19, 17, 7)	$\Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{17}$	(23, 21, 13, 5)	$\Delta_{23,13} \oplus \Delta_{21,5}$
(21, 19, 17, 15)	$\Delta_{21} \oplus \Delta_{17}[3]$	(23, 21, 13, 9)	$\Delta_{23,13} \oplus \Delta_{21,9}$
(23, 5, 3, 1)	$\Delta_{23}^2 \oplus [6]$	(23, 21, 15, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{15} \oplus [2]$
(23, 9, 3, 1)	$\Delta_{23,9} \oplus [4]$	(23, 21, 15, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{15}$
(23, 13, 3, 1)	$\Delta_{23,13} \oplus [4]$	(23, 21, 15, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{15}$
(23, 13, 11, 1)	$\Delta_{23,13} \oplus \Delta_{11} \oplus [2]$	(23, 21, 15, 13)	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{15}$
(23, 13, 11, 5)	$\Delta_{23,13,5} \oplus \Delta_{11}$	(23, 21, 17, 1)	$\mathrm{Sym}^2 \Delta_{11}[2] \oplus \Delta_{17}$
(23, 13, 11, 9)	$\Delta_{23}^2 \oplus \Delta_{11}[3]$	(23, 21, 17, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{17}$
(23, 15, 11, 3)	$\Delta_{23,15,3} \oplus \Delta_{11}$	(23, 21, 17, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{17}$
(23, 15, 11, 7)	$\Delta_{23,15,7} \oplus \Delta_{11}$	(23, 21, 17, 13)	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{17}$
(23, 17, 3, 1)	$\Delta_{23}^2 \oplus \Delta_{17} \oplus [4]$	(23, 21, 19, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus [2]$
(23, 17, 7, 1)	$\Delta_{23,7} \oplus \Delta_{17} \oplus [2]$	(23, 21, 19, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{19}$
(23, 17, 11, 1)	$\Delta_{23}^2 \oplus \Delta_{17} \oplus \Delta_{11} \oplus [2]$	(23, 21, 19, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{19}$
(23, 17, 11, 5)	$\Delta_{23,17,5} \oplus \Delta_{11}$	(23, 21, 19, 13)	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{19}$
(23, 17, 11, 9)	$\Delta_{23,17,9} \oplus \Delta_{11}$	(23, 21, 19, 17)	$\Delta_{23}^2 \oplus \Delta_{19}[3]$
(23, 17, 15, 1)	$\Delta_{23}^2 \oplus \Delta_{17} \oplus \Delta_{15} \oplus [2]$	(25, 7, 3, 1)	$\Delta_{25,7} \oplus [4]$

12. THE 121 LEVEL 1 AUTOMORPHIC REPRESENTATIONS OF $\mathrm{SO}(25)$ WITH TRIVIAL COEFFICIENTS

	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{11}[9] \oplus [2]$
[24]	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{15}[5] \oplus [10]$
$\Delta_{15}[9] \oplus [6]$	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus [14]$
$\Delta_{17}[7] \oplus [10]$	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus [18]$
$\Delta_{19}[5] \oplus [14]$	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{15}[5] \oplus [8]$
$\Delta_{21}[3] \oplus [18]$	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{17}[3] \oplus [12]$
$\Delta_{23}^2 \oplus [22]$	$\Delta_{23,7} \oplus \Delta_{21,9} \oplus \Delta_{15}[5] \oplus [6]$
$\Delta_{23}^2 \oplus \Delta_{11}[11]$	$\Delta_{23,9} \oplus \Delta_{17}[5] \oplus \Delta_{11} \oplus [8]$
$\mathrm{Sym}^2 \Delta_{11}[2] \oplus \Delta_{11}[9]$	$\Delta_{23,9} \oplus \Delta_{21} \oplus \Delta_{15}[5] \oplus [8]$
$\Delta_{19}[5] \oplus \Delta_{11}[3] \oplus [8]$	$\Delta_{23,13} \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus [12]$
$\Delta_{21}[3] \oplus \Delta_{11}[7] \oplus [4]$	$\Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus [12]$
$\Delta_{21}[3] \oplus \Delta_{15}[3] \oplus [12]$	$\Delta_{23,19,3} \oplus \Delta_{21} \oplus \Delta_{11}[7] \oplus [2]$
$\Delta_{21}[3] \oplus \Delta_{17} \oplus [16]$	$\Delta_{23,19,11} \oplus \Delta_{21} \oplus \Delta_{15}[3] \oplus [10]$
$\Delta_{23}^2 \oplus \Delta_{15}[7] \oplus [8]$	$\Delta_{23,19,11} \oplus \Delta_{21,9} \oplus \Delta_{15}[3] \oplus [8]$
$\Delta_{23}^2 \oplus \Delta_{17}[5] \oplus [12]$	$\Delta_{23,15,7} \oplus \Delta_{19}[3] \oplus \Delta_{11}[3] \oplus [6]$
$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus [16]$	$\Delta_{21}[3] \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [8]$
$\Delta_{23}^2 \oplus \Delta_{21} \oplus [20]$	$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [8]$
$\Delta_{21,9}[3] \oplus \Delta_{15}[3] \oplus [6]$	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus \Delta_{11}[3] \oplus [8]$
$\Delta_{21,13}[3] \oplus \Delta_{17} \oplus [10]$	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{11}[7] \oplus [4]$
$\Delta_{23,7} \oplus \Delta_{15}[7] \oplus [6]$	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus [12]$
$\Delta_{21}[3] \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [10]$	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus [16]$
$\Delta_{21}[3] \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [6]$	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [10]$
$\Delta_{21}[3] \oplus \Delta_{17} \oplus \Delta_{15} \oplus [14]$	
$\Delta_{23}^2 \oplus \Delta_{17}[5] \oplus \Delta_{11} \oplus [10]$	
$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus \Delta_{11}[5] \oplus [6]$	
$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus [14]$	

$$\begin{aligned}
& \Delta_{21,5}[3] \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [2] & \Delta_{23,17,9} \oplus \Delta_{21,13} \oplus \Delta_{19,7} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [6] \\
& \Delta_{23,7} \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [6] & \Delta_{23,15,3} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [2] \\
& \Delta_{23,7} \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus \Delta_{11}[3] \oplus [6] & \Delta_{23,15,7} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [6] \\
& \Delta_{23,7} \oplus \Delta_{21,5} \oplus \Delta_{17}[3] \oplus \Delta_{11}[3] \oplus [4] & \Delta_{23,15,7} \oplus \Delta_{21,5} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [4] \\
& \Delta_{23,9} \oplus \Delta_{21,13} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [8] & \Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [8] \\
& \Delta_{23,13} \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus \Delta_{11} \oplus [10] & \Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [6] \\
& \Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [10] & \Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [4] \\
& \Delta_{23,13} \oplus \Delta_{19,7}[3] \oplus \Delta_{15} \oplus \Delta_{11} \oplus [4] & \Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [10] \\
& \Delta_{23,13} \oplus \Delta_{21,9} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [8] & \Delta_{23,7} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [6] \\
& \Delta_{23,17,5} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{11}[5] \oplus [4] & \Delta_{23,7} \oplus \Delta_{21,5} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [4] \\
& \Delta_{23,19,3} \oplus \Delta_{21,5} \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [2] & \Delta_{23,9} \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [8] \\
& \Delta_{23,19,11} \oplus \Delta_{21,13} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [10] & \Delta_{23,9} \oplus \Delta_{21,13} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [6] \\
& \Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [10] & \Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [10] \\
& \Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [6] & \Delta_{23,13} \oplus \Delta_{21,9} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [8] \\
& \Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [14] & \Delta_{23,13} \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [6] \\
& \Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [4] & \Delta_{23,13,5} \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [4] \\
& \Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [8] & \\
& \Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [6] & \\
& \Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [12] & \\
& \Delta_{23,7} \oplus \Delta_{21,9} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [6] & \\
& \Delta_{23,9} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [8] & \\
& \Delta_{23,9} \oplus \Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [6] & \\
& \Delta_{23,9} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [4] & \\
& \Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [12] & \\
& \Delta_{23,17,5} \oplus \Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [4] & \\
& \Delta_{23,17,9} \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [8] &
\end{aligned}$$

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